# APPENDIX

Let $\circ$ denote a non-$\bullet$ result. Let $\mathbb{C} = \mathsf{dom}(\tau)$.

## Must-Analysis

**Lemma A.1.** *For all $\Gamma_1$, $\Gamma_2$, if $\vdash_{\mathsf{dep}} \Gamma_1$ and $\vdash_{\mathsf{dep}} \Gamma_2$, then $\vdash_{\mathsf{dep}} \Gamma_1 \odot \Gamma_2$.*

*Proof: Assume the opposite. Then, for some $C'$ and $C$, $C' \in \mathsf{dep}(C)$, $\Gamma(C) = \mathrm{I}$, and $\Gamma(C') = \bullet$, where $\Gamma = \Gamma_1 \odot \Gamma_2$. By definition of $\odot$, $\Gamma_1(C) = \Gamma_2(C) = \mathrm{I}$. But then, since $\vdash_{\mathsf{dep}} \Gamma_j$, $\Gamma_j(C'') = \mathrm{I}$ for all $C'' \in \mathsf{dep}(C)$. Therefore $\Gamma(C'') = \mathrm{I}$ for all $C'' \in \mathsf{dep}(C)$ (including in particular $C'$) by definition of $\odot$, a contradiction.* □

**Lemma 7.3** (dep-consistency preservation) *For all $\Gamma$, $\Gamma'$, $t$, $pc$ and $\ell'$, if*

*i)* $pc \vdash \Gamma\,\{t\}\,\Gamma' : \ell'$
*ii)* $\vdash_{\mathsf{dep}} \Gamma$

*then*

*iii)* $\vdash_{\mathsf{dep}} \Gamma'$,
*iv)* $\forall C\,.\,C \in \mathsf{dep}(t) \implies \Gamma'^{\mathsf{s}}(C) = \mathrm{I}$, *if $t = a$ for some $a$*
*v)* $\forall C\,.\,\Gamma'^{\mathsf{s}}(C) \neq \Gamma^{\mathsf{s}}(C) \implies C \in \mathsf{dep}(t)$, *if $t = a$ for some $a$*

*holds.*

*Proof for $a$:* By mutual induction in the height $j$ of the typing derivation of each of $e$, $c$ and $i$.

**Base $e$:** Two cases to consider.

**(NUM$_\vdash$)** : Then $e = n$ for some $n$.
$\Gamma' = \Gamma$, so *iii)* holds.
$\mathsf{dep}(n) = \emptyset$, so *iv)* and *v)* hold.

**(VAR$_\vdash$)** : Same argument as in (NUM$_\vdash$) case.

**Base $c$:** Two cases to consider.

**(INIT-T$_\vdash$)** : Then $e = C\,\{i\} = \tau(C)$.
$\Gamma' = \Gamma$, so *iii)* holds.
From *ii)* and $\Gamma(C) = \mathrm{I}$, we get that *iv)* and *v)* hold.

**(INIT-S-T$_\vdash$)** : Same argument as in (INIT-T$_\vdash$) case.

**Base $i$:** No case to consider.

We now assume Lemma 7.3 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.3 holds for $e$, $c$ and $i$ with typing derivation height $n + 1$.

**Inductive step $e$:** Three cases to consider.

**(FIELD$_\vdash$)** : Then $e = C.x$ for some $C.x$.
By *i)*, $pc \vdash \Gamma\,\{\tau(C)\}\,\Gamma' : \ell'$.
By (IH), *iii)* holds.
Also, $\mathsf{dep}(C.x) = \mathsf{dep}(\tau(C))$.
So by (IH), *iv)* and *v)* holds.

**(OP-T$_\vdash$)** : Then $e = e_1 \oplus_{\mathsf{T}} e_2$ for some $e_1$, $e_2$ and operator $\oplus_{\mathsf{T}}$.
With $\ell' = \ell_1 \sqcup \ell_2$, by *i)*, $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma_1$ holds.
By *i)*, $pc \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma'$ holds.
So *iii)* holds.
By (IH), $\forall C \in \mathsf{dep}(e_1)\,.\,\Gamma_1(C) = \mathrm{I}$.
Also by (IH), $\forall C\,.\,{\Gamma_1}^{\mathsf{s}}(C) \neq \Gamma^{\mathsf{s}}(C) \implies C \in \mathsf{dep}(e_1)$.
From Lemma 7.2, we get $\forall C \in \mathsf{dep}(e_1)\,.\,\Gamma'(C) = \mathrm{I}$.
By (IH), $\forall C \in \mathsf{dep}(e_2)\,.\,\Gamma_1(C) = \mathrm{I}$.
Also by (IH), $\forall C\,.\,\Gamma'^{\mathsf{s}}(C) \neq {\Gamma_1}^{\mathsf{s}}(C) \implies C \in \mathsf{dep}(e_2)$.
Since $\mathsf{dep}(e) = \mathsf{dep}(e_1) \cup \mathsf{dep}(e_2)$, *iv)* and *v)* hold.

**(OP-P$_\vdash$)** : Near-identical argument as in (OP-T$_\vdash$) case.

**Inductive step $c$:** Three cases to consider.

**(INIT-F$_\vdash$)** : Then $c = C\,\{i\} = \tau(C)$.
By *i)*, $pc \sqcup \Gamma^{\mathsf{c}}(C) \sqcup \Gamma^{\mathsf{e}}(C) \vdash \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]\,\{i\}\,\Gamma'[C \mapsto^{\mathsf{s}} \mathrm{B}] : \ell_i$, where $\ell' = \ell_i \sqcup \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}]$.
Here, $\Gamma' = \Gamma''[C \mapsto \langle \mathrm{I}, pc, \ell_i \rangle]$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}]$.

Also by (IH), $\forall \hat{C} \in \mathsf{dep}(i)\,.\,\Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}](\hat{C}) = \mathrm{I}$.
Also by (IH), $\forall C\,.\,\Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}]^{\mathsf{s}}(C) \neq \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \implies C \in \mathsf{dep}(i)$.
By the definition of $\Gamma'$, *iii)*, *iv)* and *v)* hold.

**(INIT-S-FT$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**(INIT-S-FF$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**Inductive step** $i$**:** One case to consider.

**(INIT$_\vdash$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$.
By *i)*, $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1}\,\{e_q\}\,\Gamma_q : \ell_q$ for all $q$ from 1 to $k$.
Induction in $k$.

**Base:** Here, $k = 0$, so $i = \texttt{skip}$.
*iii)*, *iv)* and *v)* follow since $\Gamma' = \Gamma$ and $\mathsf{dep}(i) = \emptyset$.

**Inductive step:** Assume Lemma 7.3 holds for $k \leq m$. This is our induction hypothesis (IH)$_k$. We must show that Lemma 7.3 holds for $k = m + 1$.
By (IH)$_k$, $\vdash_{\mathsf{dep}} \Gamma_m$.
Now, $\Gamma_{m+1} = \Gamma'$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma'$. So *iii)* holds.
By (IH)$_k$, $\forall \hat{C} \in \mathsf{dep}(C.x_1 := e_1; \ldots; C.x_k := e_k)\,.\,{\Gamma_m}^{\mathsf{s}}(\hat{C}) = \mathrm{I}$.
Also, $\forall \hat{C}\,.\,\Gamma_m[C \mapsto^{\mathsf{s}} \mathrm{B}]^{\mathsf{s}}(\hat{C}) \neq \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]^{\mathsf{s}}(\hat{C}) \implies$
$C \in \mathsf{dep}(C.x_1 := e_1; \ldots; C.x_k := e_k)$.
By (IH), $\forall \hat{C} \in \mathsf{dep}(e_{m+1})\,.\,{\Gamma_{m+1}}^{\mathsf{s}}(\hat{C}) = \mathrm{I}$.
Also, $\forall \hat{C}\,.\,\Gamma_{m+1}[C \mapsto^{\mathsf{s}} \mathrm{B}]^{\mathsf{s}}(\hat{C}) \neq \Gamma_m[C \mapsto^{\mathsf{s}} \mathrm{B}]^{\mathsf{s}}(\hat{C}) \implies$
$C \in \mathsf{dep}(e_{m+1}) \ldots; C.x_k := e_k$.
Since $\mathsf{dep}(i) = \mathsf{dep}(C.x_1 := e_1; \ldots; C.x_k := e_k) \cup \mathsf{dep}(C.x_{m+1} := e_{m+1})$, $\mathsf{dep}(C.x_{m+1} := e_{m+1}) = \mathsf{dep}(e_{m+1})$, and $\Gamma_m \sqsubseteq \Gamma_{m+1}$, *iv)* and *v)* follow from (IH).

□

*Proof sketch for s:* As for $a$, the proof for $s$ is by induction in the height $j$ of the typing derivation of $s$. For $s$, all we need in the future is to be certain that invariant $\vdash_{\mathsf{dep}} \Gamma$ is preserved as the type system threads $\Gamma$ through statements, that is, that *iii)* holds. The definition of $\mathsf{dep}(\cdot)$ could be extended to a homomorphism on $s$, in which case *v)* would hold for $s$. However, *iv)* will not hold for $s$, as evidenced for instance by $\texttt{if}\ h\ \texttt{then}\ C.x := 0\ \texttt{else skip}$; this $s$ depends on $C$, but $C$ is not necessarily initialized after a successful run of $s$. The cultprit is the $\odot$ operator — the only means by which $\Gamma$ changes in the statement typing rules. Thus, the (IF$_\vdash$), (TRY$_\vdash$) and (WHILE$_\vdash$), the only rules using the $\odot$ operator, constitute the only interesting cases in this proof. As the semantics and typing of `while` statements can be viewed as a combination of sequential composition and `if` statements, the interesting cases become the (IF$_\vdash$), (TRY$_\vdash$) and (SEQ$_\vdash$) rules, which we prove here.

Assume Lemma 7.3 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.3 holds for $s$ with typing derivation height $n + 1$.

**Inductive step:** Three (interesting) cases to consider.

**(SEQ$_\vdash$):** Then $s = s_1; s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma\,\{s_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1\,\{s_2\}\,\Gamma' : \ell_2$, where $\ell' = \ell_1 \sqcup \ell_2$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma_1$.
Again by (IH), $\vdash_{\mathsf{dep}} \Gamma'$.
So *iii)* holds.

**(IF$_\vdash$):** Then $s = \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2$, for some $e$, $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma\,\{e\}\,\hat{\Gamma} : \hat{\ell}$, $pc \sqcup \hat{\ell} \sqcup \mathsf{lvl}(e) \vdash \hat{\Gamma}\,\{s_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \hat{\ell} \sqcup \mathsf{lvl}(e) \vdash \hat{\Gamma}\,\{s_2\}\,\Gamma_2 : \ell_2$, where $\ell' = \ell_1 \sqcup \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
By Lemma 7.3 for $a$, $\vdash_{\mathsf{dep}} \hat{\Gamma}$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma_1$ and $\vdash_{\mathsf{dep}} \Gamma_2$.
By Lemma A.1, $\vdash_{\mathsf{dep}} \Gamma'$.
So *iii)* holds.

**(TRY$_\vdash$):** Then $s = \texttt{try}\ s_1\ \texttt{catch}\ s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma\,\{s_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma \odot \Gamma_1\,\{s_2\}\,\Gamma_2 : \ell_2$, where $\ell' = \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma_1$.
By Lemma A.1, $\vdash_{\mathsf{dep}} \Gamma \odot \Gamma_1$.
By (IH), $\vdash_{\mathsf{dep}} \Gamma_2$.
By Lemma A.1, $\vdash_{\mathsf{dep}} \Gamma_1 \odot \Gamma_2$.
So *iii)* holds.

□

It is easy to see, by investigating the type resp. semantics rules that introduce and eliminate a $C \mapsto \text{B}$ into $\Gamma^{\text{s}}$ resp. $\sigma$, that the following two lemmas hold.

**Lemma A.2.** *For all* $\Gamma$, $\Gamma'$, $t$, *if* $\_ \vdash \Gamma\,\{t\}\,\Gamma' : \_$, *then for all* $C$, $\Gamma^{\text{s}}(C) = \text{B} \iff \Gamma'^{\text{s}}(C) = \text{B}$.

**Lemma A.3.** *For all* $\sigma$, $\sigma'$, $t$, *if* $\langle \sigma, t\rangle \Rightarrow \langle \sigma', \_\rangle$, *then for all* $C$, $\sigma(C) = \text{B} \iff \sigma'(C) = \text{B}$.

**Lemma 7.4** (agreement preservation) *For all* $\Gamma$, $\Gamma'$, $t$, $pc$ *and* $\ell'$, *if*

*i)* $pc \vdash \Gamma\,\{t\}\,\Gamma' : \ell'$
*ii)* $\vdash_{\mathsf{dep}} \Gamma$, $\vdash_{\mathsf{dep}} \sigma$, $\Gamma \models_{\mathsf{dep}} \sigma$
*iii)* $\langle \sigma, t\rangle \Rightarrow \langle \sigma', R\rangle$

*then*

*iv)* $\vdash_{\mathsf{dep}} \sigma'$
*v)* $R \neq \bullet \implies \Gamma' \models_{\mathsf{dep}} \sigma'$

*holds.*

*Proof for a:* By mutual induction in the height $j$ of the typing derivation of each of $e$, $c$ and $i$.

**Base** $e$**:** Two cases to consider.

**(**NUM$_\vdash$**)** : Then $e = n$ for some $n$.
Only (NUM$_\Rightarrow$) can establish *iii)*.
By this rule, $\sigma' = \sigma$.
So *iv)* holds.
By (NUM$_\vdash$), $\Gamma' = \Gamma$.
By *ii)*, $\Gamma' \models_{\mathsf{dep}} \sigma'$.
So *v)* holds.

**(**VAR$_\vdash$**)** : Same argument as in (NUM$_\vdash$) case.

**Base** $c$**:** Two cases to consider.

**(**INIT-T$_\vdash$**)** : Then $e = C\,\{i\} = \tau(C)$.
By (INIT-T$_\vdash$), $\Gamma' = \Gamma$.
By *ii)* and since $\Gamma(C) = \text{I}$, only (INIT-T$_\Rightarrow$) can conclude *iii)*.
(INIT-T$_\Rightarrow$) gives $\sigma' = \sigma$.
So *iv)* holds.
By *ii)*, $\Gamma' \models_{\mathsf{dep}} \sigma'$.
So *v)* holds.

**(**INIT-S-T$_\vdash$**)** : Same argument as in (INIT-T$_\vdash$) case.

**Base** $i$**:** No cases to consider.

We now assume Lemma 7.4 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.4 holds for $e$, $c$ and $i$ with typing derivation height $n + 1$.

**Inductive step** $e$**:** Three cases to consider.

**(**FIELD$_\vdash$**)** : Then $e = C.x$ for some field $C.x$.
Let $\langle \sigma, \tau(C)\rangle \Rightarrow \langle \sigma', I\rangle$.
By *i)*, $pc \vdash \Gamma\,\{\tau(C)\}\,\Gamma' : \ell'$.
Case on $I$.

$I = \bullet$**:** Then (FIELD-E$_\Rightarrow$) was used to establish *iii)*.
By (FIELD-E$_\Rightarrow$), $R = \bullet$.
So *v)* holds vacuously.
By (IH), $\vdash_{\mathsf{dep}} \sigma'$.
So *iv)* holds.

$I = \text{I}$**:** Then (FIELD-OK$_\Rightarrow$) was used to establish *iii)*.
By (FIELD-OK$_\Rightarrow$), $R = \sigma'(C.x)$.
By (IH), $\vdash_{\mathsf{dep}} \sigma'$ and $\Gamma' \models_{\mathsf{dep}} \sigma'$.
So *iv)* and *v)* hold.

**(**OP-T$_\vdash$**)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some total operator $\oplus$.
By *i)* we have $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell' = \ell_1 \sqcup \ell_2$.
Case on the rule used to establish *iii)*.

**(**OP-EL$_\Rightarrow$**):** Then $\langle \sigma, e_1\rangle \Rightarrow \langle \sigma', \bullet\rangle$.
Since $R = \bullet$, *v)* holds vacuously.

By (IH), *v)* holds.

**(OP-ER$_\Rightarrow$):** Then $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, n_1 \rangle$ and $\langle \sigma_1, e_2 \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
Since $R = \bullet$, *v)* holds vacuously.
By Lemma 7.3 and by (IH), $\vdash_{\mathsf{dep}} \Gamma_1$, $\vdash_{\mathsf{dep}} \sigma_1$ and $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
By (IH), *iv)* holds.

**(OP-OK$_\Rightarrow$):** Then $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, n_1 \rangle$ and $\langle \sigma_1, e_2 \rangle \Rightarrow \langle \sigma', n_2 \rangle$, where $n = n_1 \oplus n_2$.
By Lemma 7.3 and by (IH), $\vdash_{\mathsf{dep}} \Gamma_1$, $\vdash_{\mathsf{dep}} \sigma_1$ and $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
By (IH), *iv)* and *v)* holds.

**(OP-P$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some partial operator $\oplus$. Case on the rule used to establish *iii)*. All cases and proofs thereof are the same as for the (OP-T$_\vdash$) case, except that (OP-P$_\vdash$) has the following additional case:

**(OP-EP$_\Rightarrow$):** As the proof of the (OP-OK$_\Rightarrow$) case, except *v)* holds vacuously.

**Inductive step** $c$**:** Three cases to consider.

**(INIT-F$_\vdash$)** : Then $c = C\ \{i\}$ for some $C$ and $i$.
Also, $\Gamma(C) = \mathrm{U}$.
By *i)* we have $pc \sqcup \Gamma^{\mathsf{e}}(C) \vdash \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]\ \{i\}\ \Gamma'' : \ell_C$,
where $\Gamma' = \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{I}, C \mapsto^{\mathsf{e}} \ell_C]$ and $\ell' = \ell_C \sqcup \Gamma''^{\mathsf{e}}(C)$.
Case on the rule used to establish *iii)*.

**(INIT-A$_\Rightarrow$)** : Then $\langle \sigma, c \rangle \Rightarrow \langle \sigma', \sigma(C) \rangle$, with $\sigma' = \sigma$.
So *iv)* holds.
Case on $\sigma(C)$.
$\sigma(C) = \bullet$ : But then $\langle \sigma, c \rangle \Rightarrow \langle \sigma', \bullet \rangle$, so *v)* holds vacuously.
$\sigma(C) = \mathrm{B}$ : But $\Gamma(C) = \mathrm{U}$, contradicting $\Gamma \models_{\mathsf{dep}} \sigma$, and thus *ii)*.
$\sigma(C) = \mathrm{I}$ : From $\vdash_{\mathsf{dep}} \sigma$, we have $\forall \hat{C} \in \mathsf{dep}(C)\,.\,\sigma(\hat{C}) = \mathrm{I}$.
By Lemma 7.3,
we have $\forall \hat{C}\,.\,\hat{C} \in \mathsf{dep}(C) \implies \Gamma'^{\mathsf{s}}(\hat{C}) = \mathrm{I}$
and $\forall \hat{C}\,.\,\Gamma'^{\mathsf{s}}(\hat{C}) \neq \Gamma^{\mathsf{s}}(\hat{C}) \implies \hat{C} \in \mathsf{dep}(C)$.
This, together with $\Gamma \models_{\mathsf{dep}} \sigma$, gives $\Gamma' \models_{\mathsf{dep}} \sigma$. Since $\sigma' = \sigma$, *v)* holds.

**(INIT-U$_\Rightarrow$)** : Let $\langle \sigma[C \mapsto \mathrm{B}], i \rangle \Rightarrow \langle \sigma'', T \rangle$, where $\sigma' = \sigma''[C \mapsto I(T)]$.
We have $\vdash_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$, $\vdash_{\mathsf{dep}} \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]$ and $\Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \models_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$.
By (IH), $\vdash_{\mathsf{dep}} \sigma''$, and $T \neq \bullet \implies \Gamma'' \models_{\mathsf{dep}} \sigma''$.
By Lemma 7.3,
we have $\forall \hat{C}\,.\,\hat{C} \in \mathsf{dep}(i) \implies \Gamma''^{\mathsf{s}}(\hat{C}) = \mathrm{I}$
and $\forall \hat{C}\,.\,\Gamma''^{\mathsf{s}}(\hat{C}) \neq \Gamma^{\mathsf{s}}(\hat{C}) \implies \hat{C} \in \mathsf{dep}(i)$.
From the definition of $\Gamma'$ and $\sigma'$,
and from $\mathsf{dep}(c) = \mathsf{dep}(i) \cup \{C\}$.
*iv)* and *v)* follow.

**(INIT-S-FT$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**(INIT-S-FF$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**Inductive step** $i$**:** One case to consider.

**(INIT$_\vdash$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$. Induction in $k$.

**Base:** Here, $k = 0$. Then $i = \texttt{skip}$.
By (INIT$_\vdash$), $\Gamma' = \Gamma$.
Only (SKIP$_\Rightarrow$) can conclude *iii)*.
(SKIP$_\Rightarrow$) gives $\sigma' = \sigma$. So *iv)* holds.
By *ii)*, $\Gamma' \models_{\mathsf{dep}} \sigma'$.
So *v)* holds.

**Inductive step:** Assume Lemma 7.4 holds for $k \leq m$. This is our induction hypothesis (IH)$_k$. We must show that Lemma 7.4 holds for $k = m + 1$.
By assumption *i)*,
we get that $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1}\ \{e_q\}\ \Gamma_q : \ell_q$,
for all $q$ from 1 to $m + 1$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_m$.
Let $\langle \sigma, C.x_1 := e_1; \ldots; C.x_m := e_m \rangle \Rightarrow \langle \sigma_m, T_m \rangle$.
By (IH)$_k$, $\vdash_{\mathsf{dep}} \sigma_m$ and $T_m \neq \bullet \implies \Gamma_m \models_{\mathsf{dep}} \sigma_m$.
Case on the rule used to establish *iii)*.

**(SEQ-E$_{\Rightarrow}$)** : Then $R = T_m = \bullet$ and $\sigma' = \sigma_m$.
So *v)* holds vacuously, and *iv)* holds.

**(SEQ-OK$_{\Rightarrow}$)** : Then $T_m = \texttt{skip}$,
and $\langle \sigma_m, C.x_{m+1} := e_{m+1} \rangle \Rightarrow \langle \sigma', R \rangle$. (*)
Two candidate rules for establishing (*); (E-E$_{\Rightarrow}$) and (FIELD-A-OK$_{\Rightarrow}$).
Both rules start by evaluating $e_{m+1}$ under $\sigma_m$.
Let $\langle \sigma_m, e_{m+1} \rangle \Rightarrow \langle \sigma_{m+1}, V_{m+1} \rangle$.
By (IH), $\vdash_{\mathsf{dep}} \sigma_{m+1}$ and $V_{m+1} \neq \bullet \implies \Gamma_{m+1} \models_{\mathsf{dep}} \sigma_{m+1}$.
Case on $V_{m+1}$.

$V_{m+1} = \bullet$**:** Then (E-E$_{\Rightarrow}$) was used to establish (*).
By (E-E$_{\Rightarrow}$), $\sigma' = \sigma_{m+1}$, and $R = \bullet$.
So *v)* holds vacuously, and *iv)* holds.

$V_{m+1} = n_{m+1}$**:** Then (FIELD-A-OK$_{\Rightarrow}$) was used to establish (*).
By (FIELD-A-OK$_{\Rightarrow}$), $\sigma' = \sigma_{m+1}[C.x_{m+1} \mapsto n_{m+1}]$ and $R = \texttt{skip}$.
Since $\sigma_{m+1}$ and $\sigma'$ do not differ in class initialization statues,
and since $\Gamma' = \Gamma_{m+1}$,
$\vdash_{\mathsf{dep}} \sigma'$ and $\Gamma' \models_{\mathsf{dep}} \sigma'$.
So *v)* and *iv)* hold.

□

*Proof sketch for s:* As for $a$, the proof for $s$ is by induction in the height $j$ of the typing derivation of $s$. As in Lemma 7.3, we prove only the interesting cases.

Assume Lemma 7.4 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.4 holds for $s$ with typing derivation height $n + 1$.

**Inductive step:** Three (interesting) cases to consider.

**(SEQ$_{\vdash}$):** Then $s = s_1; s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma \{s_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1 \{s_2\} \Gamma' : \ell_2$, where $\ell' = \ell_1 \sqcup \ell_2$.
Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\vdash_{\mathsf{dep}} \sigma_1$ and $T_1 \neq \bullet \implies \Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
Case on $T_1$.

$T_1 = \bullet$**:** Then $\sigma' = \sigma_1$ and $R = \bullet$.
So *v)* holds vacuously, and *iv)* holds.

$T_2 = \texttt{skip}$**:** Then $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_1$.
By (IH), $\vdash_{\mathsf{dep}} \sigma'$ and $R \neq \bullet \implies \Gamma' \models_{\mathsf{dep}} \sigma'$.
So *v)* and *iv)* hold.

**(IF$_{\vdash}$):** Then $s = \texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2$, for some $e$, $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma \{e\} \hat{\Gamma} : \hat{\ell}$, $pc \sqcup \hat{\ell} \sqcup \mathsf{lvl}(e) \vdash \hat{\Gamma} \{s_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \hat{\ell} \sqcup \mathsf{lvl}(e) \vdash \hat{\Gamma} \{s_2\} \Gamma_2 : \ell_2$, where $\ell' = \ell_1 \sqcup \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
Let $\langle \sigma, e \rangle \Rightarrow \langle \sigma_e, V \rangle$.
By Lemma 7.4 for $a$, $\vdash_{\mathsf{dep}} \sigma_e$ and $\hat{\Gamma} \models_{\mathsf{dep}} \sigma_e$.
Case on $V$.

$V = \bullet$**:** Then $\sigma' = \sigma_e$ and $R = \bullet$.
So *v)* holds vacuously, and *iv)* holds.

$V = n$**:** Assume $n = 0$ (argument for $n = \bar{0}$ near-identical).
Then $\langle \sigma_e, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By (IH), $\vdash_{\mathsf{dep}} \sigma'$ and $R \neq \bullet \implies \Gamma_2 \models_{\mathsf{dep}} \sigma'$. (*)
So *iv)* holds.
It remains to be shown that $R \neq \bullet \implies \Gamma_1 \odot \Gamma_2 \models_{\mathsf{dep}} \sigma'$.
By Lemma A.2, we get that $\hat{\Gamma}^{\mathsf{s}}(C) = \mathtt{B} \iff {\Gamma_j}^{\mathsf{s}}(C) = \mathtt{B}$.
Thus, by transitivity of " $\iff$ ", ${\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B} \iff {\Gamma_2}^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$.
By definition of $\odot$, ${\Gamma_2}^{\mathsf{s}}(C) = \mathtt{B} \iff {\Gamma'}^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$.
By (*), Pt. 2) of Definition 7.1 for $\Gamma' \models_{\mathsf{dep}} \sigma'$ is satisfied.
By (*) and since ${\Gamma'}^{\mathsf{s}} \sqsubseteq \Gamma_2^{\mathsf{s}}$, Pt. 1) of Definition 7.1 for $\Gamma' \models_{\mathsf{dep}} \sigma'$ is satisfied.
So *v)* holds.

**(TRY$_{\vdash}$):** Then $s = \texttt{try } s_1 \texttt{ catch } s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma \{s_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma \odot \Gamma_1 \{s_2\} \Gamma_2 : \ell_2$,

where $\ell' = \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\vdash_{\mathsf{dep}} \sigma_1$ and $T_1 \neq \bullet \implies \Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
By Lemma A.2, we get that ${\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B} \iff \Gamma^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$.
By definition of $\odot$, ${\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B} \iff \Gamma \odot {\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$.
By Lemma A.2 again, we get that $\Gamma \odot {\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B} \iff {\Gamma_2}^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$.
By transitivity of " $\iff$ ", ${\Gamma_1}^{\mathsf{s}}(C) = \mathtt{B} \iff {\Gamma_2}^{\mathsf{s}}(C) = \mathtt{B}$, for all $C$. (*)
By definition of $\odot$, ${\Gamma'}^{\mathsf{s}} \sqsubseteq \Gamma_1^{\mathsf{s}}$.
Thus $\Gamma' \models_{\mathsf{dep}} \sigma_1$.
Case on $T_1$.

$T_1 = \mathtt{skip}$**:** Then $\sigma_1 = \sigma'$ and $R = T_1$.
So *iv)* and *v)* hold.

$T_1 = \bullet$**:** Then $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By Lemma 7.3, since $(\Gamma \odot \Gamma_1)^{\mathsf{s}} = \Gamma^{\mathsf{s}}$, $\vdash_{\mathsf{dep}} \Gamma \odot \Gamma_1$.
By (IH), $\vdash_{\mathsf{dep}} \sigma'$ and $R \neq \bullet \implies \Gamma_2 \models_{\mathsf{dep}} \sigma'$.
So *iv)* holds.
By definition of $\odot$, ${\Gamma'}^{\mathsf{s}} \sqsubseteq \Gamma_2^{\mathsf{s}}$.
This, and (*), gives $\Gamma' \models_{\mathsf{dep}} \sigma_2$.
So *v)* holds.

□

## Errors

**Lemma 7.5** (error consistency preservation) *For all $\sigma$, $\sigma'$ and $t$, if*

*i)* $\vdash_{\mathsf{err}} \sigma$
*ii)* $\langle \sigma, t \rangle \Rightarrow \langle \sigma', R \rangle$

*then*

*iv)* $\vdash_{\mathsf{err}} \sigma'$

*holds.*

*Proof for $a$:* By mutual induction in the height $j$ of the reduction derivation of each of $e$, $c$ and $i$.

**Base $e$:** Two cases to consider.

**(**NUM$_{\Rightarrow}$**)** : Then $e = n$ for some $n$.
As $\sigma' = \sigma$, *iv)* follows.

**(**VAR$_{\Rightarrow}$**)** : Same argument as in (NUM$_{\vdash}$) case.

**Base $c$:** Three cases to consider.

**(**INIT-A$_{\vdash}$**)** : Then $e = C\,\{i\} = \tau(C)$.
As $\sigma' = \sigma$, *iv)* follows.

**(**INIT-S-A$_{\Rightarrow}$**)** : Similar argument as in (INIT-A$_{\Rightarrow}$) case.

**(**INIT-S-UF$_{\Rightarrow}$**)** : Then by *i)* we get

$$\forall \hat{C} \neq C \,.\, \sigma'(\hat{C}) = \bullet \implies \exists \sigma''; (\sigma'' \sqsubseteq \sigma'), (\vdash_{\mathsf{err}} \sigma''), (\sigma''(\hat{C}) \neq \bullet) \,.\, \langle \sigma'', \tau(\hat{C}) \rangle \Rightarrow \langle \_, \bullet \rangle.$$

We have $\sigma' = \sigma[C \mapsto \bullet]$. So $\sigma'(C) = \bullet$. Since $\sigma'$ and $\sigma$ are so similar, to prove *iv)* we must only show that

$$\exists \sigma''; (\sigma'' \sqsubseteq \sigma'), (\vdash_{\mathsf{err}} \sigma''), (\sigma''(C) \neq \bullet) \,.\, \langle \sigma'', \tau(C) \rangle \Rightarrow \langle \_, \bullet \rangle.$$

This $\sigma''$ is $\sigma$; by Lemma 7.1, $\sigma \sqsubseteq \sigma'$. *iv)* follows from (INIT-S-UF$_{\Rightarrow}$) and *i)*.

**Base $i$:** One case to consider.

**(**SKIP$_{\Rightarrow}$**)** : Then $i = \mathtt{skip}$.
As $\sigma' = \sigma$, *iv)* follows.

We now assume Lemma 7.5 holds for $e$, $c$ and $i$ with reduction derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.5 holds for $e$, $c$ and $i$ with typing derivation height $n + 1$.

**Inductive step $e$:** Six cases to consider.

**(**FIELD-E$_{\Rightarrow}$**)** : Then $e = C.x$ for some $C.x$.

We have $\langle \sigma, \tau(C) \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
By (IH), *iv)* holds.

**(**FIELD-OK$_\Rightarrow$**)** Near-identical argument as in (FIELD-E$_\Rightarrow$) case.

**(**OP-EL$_\Rightarrow$**)** : Then $e = e_1 \oplus e_2$ for some operator $\oplus$.
We have $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
By (IH), *iv)* holds.

**(**OP-ER$_\Rightarrow$**)** : Then $e = e_1 \oplus e_2$ for some operator $\oplus$.
We have $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, \circ \rangle$.
By (IH), $\vdash_{\mathsf{err}} \sigma_1$.
We also have $\langle \sigma_1, e_2 \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
By (IH), *iv)* holds.

**(**OP-EP$_\Rightarrow$**)** : Near-identical argument as in (OP-ER$_\Rightarrow$) case.

**(**OP-OK$_\Rightarrow$**)** : Near-identical argument as in (OP-ER$_\Rightarrow$) case.

**Inductive step** $c$**:** Four cases to consider.

**(**INIT-U$_\Rightarrow$**)** : Then $c = C\,\{i\}$ for some $C$ and $i$.
We have $\langle \sigma[C \mapsto \text{B}], i \rangle \Rightarrow \langle \sigma''[C \mapsto \text{B}], T \rangle$, where $\sigma' = \sigma''[C \mapsto I(T)]$.
By *i)*, $\sigma(C) = \text{U}$ and Definition 7.3, $\vdash_{\mathsf{err}} \sigma[C \mapsto \text{B}]$.
By (IH), $\vdash_{\mathsf{err}} \sigma''[C \mapsto \text{B}]$. Case on $T$.
$T = \texttt{skip}$**:** Then, $I(T) = \text{I}$, so *iv)* follows by Definition 7.3 and the definition of $\sigma'$.
$T = \bullet$**:** Then $I(T) = \bullet$, so $\sigma'(C) = \bullet$.
Since $\vdash_{\mathsf{err}} \sigma''[C \mapsto \text{B}]$ and $\sigma''[C \mapsto \text{B}] \sqsubseteq \sigma'$, we have

$$\forall \hat{C} \neq C \,.\, \sigma'(\hat{C}) = \bullet \implies \exists \sigma'''; (\sigma''' \sqsubseteq \sigma'), (\vdash_{\mathsf{err}} \sigma'''), (\sigma'''(\hat{C}) \neq \bullet) \,. \\ \langle \sigma''', \tau(\hat{C}) \rangle \Rightarrow \langle \_, \bullet \rangle.$$

Since $\sigma'$ and $\sigma''[C \mapsto \text{B}]$ are so similar, to prove *iv)* we must only show that

$$\exists \sigma'''; (\sigma''' \sqsubseteq \sigma'), (\vdash_{\mathsf{err}} \sigma'''), (\sigma'''(C) \neq \bullet) \,. \\ \langle \sigma''', \tau(C) \rangle \Rightarrow \langle \_, \bullet \rangle.$$

This $\sigma'''$ is $\sigma$; by Lemma 7.1, $\sigma \sqsubseteq \sigma'$. *iv)* follows from (INIT-U$_\Rightarrow$) and *i)*.

**(**INIT-S-UI$_\Rightarrow$**), (**INIT-S-UUF$_\Rightarrow$**) and (**INIT-S-UUI$_\Rightarrow$**):** All similar in style to (INIT-U$_\Rightarrow$) case.

**Inductive step** $i$**:** Observe that $\sigma(C) = \text{B}$. Five cases to consider.

**(**E-E$_\Rightarrow$**)** : Here $i = C.x := e$ and $\langle \sigma, e \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
*iv)* follows from (IH).

**(**FIELD-A-E$_\Rightarrow$**)** : Impossible as $\sigma(C) = \text{B}$.

**(**FIELD-A$_\Rightarrow$**)** : Then $i = C.x := e$ for some $C.x$ and $e$.
We have $\langle \sigma, e \rangle \Rightarrow \langle \sigma'', n \rangle$ and $\langle \sigma'', C.x \rangle \Rightarrow \langle \sigma'', n' \rangle$, last reduction holding since $\sigma(C) = \text{B}$. Here, $\sigma' = \sigma''[C.x \mapsto n]$.
By (IH), $\vdash_{\mathsf{err}} \sigma''$, and thus *iv)*.

**(**SEQ-E$_\Rightarrow$**)** : Then $i = i_1; i_2$ for some $i_1$ and $i_2$.
We have $\langle \sigma, i_1 \rangle \Rightarrow \langle \sigma', \bullet \rangle$.
By (IH), *iv)*.

**(**SEQ-OK$_\Rightarrow$**)** : Then $i = i_1; i_2$ for some $i_1$ and $i_2$.
We have $\langle \sigma, i_1 \rangle \Rightarrow \langle \sigma_1, \circ \rangle$.
By (IH), $\vdash_{\mathsf{err}} \sigma_1$.
We also have $\langle \sigma_1, i_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By (IH), *iv)*.

□

*Proof of s:* Follows from Lemma 7.5 for $a$ and Lemma 7.1, since no semantic rule for statement evaluation performs a class initialization. To give an impression of how the proof for the various cases goes, we prove one sample case below, namely that of (TRY$_\vdash$).

Assume Lemma 7.5 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.5 holds for $s$ with typing derivation height $n + 1$.

**Inductive step:** One (interesting) case to consider.

**(**TRY$_\vdash$**)** : Then $s = \texttt{try}\ s_1\ \texttt{catch}\ s_2$, for some $s_1$ and $s_2$.

Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\vdash_{\mathsf{err}} \sigma_1$.
Case on $T_1$.
$T_1 = \texttt{skip}$**:** Then $\sigma' = \sigma_1$.
So *iv)* holds.
$T_1 = \bullet$**:** Let $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By (IH), $\vdash_{\mathsf{err}} \sigma'$.
So *iv)* holds.

□

**Lemma 7.6** (error leaks $pc$) *For all* $\Gamma$, $\Gamma'$, $t$, $pc$ *and* $\ell'$, *if*

*1)* $pc \vdash \Gamma\,\{t\}\,\Gamma' : \ell'$,
*2)* $\vdash_{\mathsf{dep}} \Gamma$,
*3)* $\Gamma \models_{\mathsf{err}} t$,

*then*

*4)* $pc \sqsubseteq \ell'$

*holds.*

*Proof for a:* By mutual induction in the height $j$ of the typing derivation of each of $e$, $c$ and $i$.

**Base** $e$**:** Two cases to consider.

**(**NUM$_\vdash$**)** : Then $e = n$ for some $n$.
Assume (towards a contradiction) that *3)* holds.
Only candidate rule for establishing *3)* is (NUM$_\Rightarrow$), and for that rule, *3)* is false, contradicting *3)*.
So *3)* is impossible.
So *4)* holds vacuously.

**(**VAR$_\vdash$**)** : Near-identical argument as in (NUM$_\vdash$) case.

**Base** $c$**:** Two cases to consider.

**(**INIT-T$_\vdash$**)** : Then $e = C\,\{i\} = \tau(C)$.
Assume (towards a contradiction) that *3)* holds.
Since $\Gamma \models_{\mathsf{dep}} \sigma$ and $\Gamma^{\mathsf{s}}(C) = \mathrm{I}$, $\sigma(C) = \mathrm{I}$, the only candidate rule to establish *3)* (INIT-A$_\Rightarrow$). For that rule, since $R = \sigma(C) = \mathrm{I}$, *3)* is false, contradicting *3)*.
So *3)* is impossible.
So *4)* holds vacuously.

**(**INIT-S-T$_\vdash$**)** : Similar argument as in (INIT-T$_\vdash$) case.

**Base** $i$**:** No cases to consider.

We now assume Lemma 7.6 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.6 holds for $e$, $c$ and $i$ with typing derivation height $n + 1$.

**Inductive step** $e$**:** Three cases to consider.

**(**FIELD$_\vdash$**)** : Then $e = C.x$ for some field $C.x$.
By *1)*, $pc \vdash \Gamma\,\{a\}\,\Gamma' : \ell'$.
Assume that *3)* holds.
Case on the candidate rules for establishing *3)*.

**(**FIELD-E$_\Rightarrow$**)** : We have $\langle \sigma, \tau(C) \rangle \Rightarrow \langle \sigma', \bullet \rangle$ and $R = \bullet$.
By (IH), with $\sigma$ as evidence that $\tau(C)$ can fail, $pc \sqsubseteq \ell'$.
So *4)* holds.

**(**FIELD-OK$_\Rightarrow$**)** : Then $R \neq \bullet$, so *3)* is false, contradicting *3)*.
So *4)* holds vacuously.

**(**OP-T$_\vdash$**)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some total operator $\oplus$.
By *1)*, $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$,
where $\ell' = \ell_1 \sqcup \ell_2$.
Assume that *3)* holds.
Case on the rule used to establish *3)*.

**(**OP-EL$_\Rightarrow$**):** Then $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma', \bullet \rangle$, with $R = \bullet$.
By (IH), with $\sigma$ as evidence that $e_1$ can fail, $pc \sqsubseteq \ell_1$.
Since $\ell_1 \sqsubseteq \ell'$, *4)* holds.

**(**OP-ER$_\Rightarrow$**):** Then $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, n_1 \rangle$ and $\langle \sigma_1, e_2 \rangle \Rightarrow \langle \sigma', \bullet \rangle$,
with $R = \bullet$.

By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_1$. By Lemma 7.4, $\vdash_{\mathsf{dep}} \sigma_1$ and $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
By Lemma 7.5, $\vdash_{\mathsf{err}} \sigma_1$.
By (IH), with $\sigma_1$ as evidence that $e_2$ can fail, $pc \sqsubseteq \ell_2$.
Since $ell_2 \sqsubseteq \ell'$, *4)* holds.

**(OP-OK$_{\Rightarrow}$):** Then $R \neq \bullet$, so *3)* is false, contradicting *3)*.
So *4)* holds vacuously.

**(OP-P$_{\vdash}$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some partial operator $\oplus$.
By *1)*, $pc \vdash \Gamma \{e_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1 \{e_2\} \Gamma' : \ell_2$,
where $\ell' = \ell_1 \sqcup \ell_2 \sqcup \mathsf{lvl}(e_1) \sqcup \mathsf{lvl}(e_2) \sqcup pc$.
*4)* follows.

**Inductive step $c$:** Three cases to consider.

**(INIT-F$_{\vdash}$)** : Then $c = C\ \{i\}$ for some $C$ and $i$.
By *1)*, $pc \sqsubseteq \Gamma^{\mathsf{c}}(C) \sqsubseteq \Gamma^{\mathsf{e}}(C) \vdash \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]\ \{i\}\ \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}] : \ell_C$,
where $\ell' = \ell_C \sqsubseteq \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}]$.
Assume that *3)* holds.
Case on $\sigma(C)$.

$\sigma(C) = \mathrm{B}$ : Impossible as $\Gamma^{\mathsf{s}}(C) = \mathrm{U}$ and $\Gamma \models_{\mathsf{dep}} \sigma$.
So *4)* holds vacuously.

$\sigma(C) = \mathrm{I}$ : Then $R \neq \bullet$, so *3)* is false, contradicting *3)*.
So *4)* holds vacuously.

$\sigma(C) = \mathrm{U}$ : Then $\langle \sigma[C \mapsto \mathrm{B}], i\rangle \Rightarrow \langle \sigma''[C \mapsto \mathrm{B}], T\rangle$,
where $\sigma' = \sigma''[C \mapsto I(T)]$.
Case on $I(T)$.

$I(T) = \mathrm{I}$ : Then $R \neq \bullet$, so *3)* is false, contradicting *3)*.
So *4)* holds vacuously.

$I(T) = \bullet$ : By Definitions 7.2 and 7.1,
$\vdash_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$ and $\Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \models_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$.
By (IH), with $\sigma[C \mapsto \mathrm{B}]$ as evidence that $i$ can fail, $\ell_C \sqsubseteq pc$.
Since $\ell_C \sqsubseteq \ell'$, *4)* follows.

$\sigma(C) = \bullet$ : By Definition 7.3, there exists some $\hat{\sigma}$ with $\hat{\sigma} \sqsubseteq \sigma$, $\vdash_{\mathsf{dep}} \hat{\sigma}$, $\Gamma \models_{\mathsf{dep}} \hat{\sigma}$ and $\hat{\sigma}(C) \neq \bullet$ for which $\langle \hat{\sigma}, c\rangle \Rightarrow \langle \hat{\sigma}', \bullet\rangle$ (*).
We already know $\hat{\sigma}(C) \neq \bullet$.
We also have $\hat{\sigma}(C) \neq \mathrm{I}$ and $\hat{\sigma}(C) \neq \mathrm{B}$, for if either were the case, (*) would have been concluded using (INIT-A$_{\Rightarrow}$), which, when $\hat{\sigma}(C) \neq \bullet$ gives $R \neq \bullet$.
So $\hat{\sigma}(C) = \mathrm{U}$. The remainder of this case equals the proof of case $\sigma(C) = \mathrm{U}$, with a hat on all the $\sigma$s.

**(INIT-S-FT$_{\vdash}$)** : Argument similar in style as in (INIT-F$_{\vdash}$) case.

**(INIT-S-FF$_{\vdash}$)** : Argument similar in style as in (INIT-F$_{\vdash}$) case.

**Inductive step $i$:** One case to consider.

**(INIT$_{\vdash}$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$.
Assume that *3)* holds.
Induction in $k$.

**Base:** Here, $k = 0$. So $i = \texttt{skip}$.
Only candidate rule for establishing *3)* is (SKIP$_{\Rightarrow}$), and for that rule, *3)* is false, contradicting *3)*.
So *3)* is impossible.
So *4)* holds vacuously.

**Inductive step:** Assume Lemma 7.6 holds for $k \leq m$. This is our induction hypothesis (IH)$_k$. We must show that Lemma 7.6 holds for $k = m + 1$.
By assumption *1)*, we get that $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1}\ \{e_q\}\ \Gamma_q : \ell_q$ for all $q$ from 1 to $m + 1$.
Let $\langle \sigma, C.x_1 := e_1; \ldots; C.x_m := e_m\rangle \Rightarrow \langle \sigma_m, T_m\rangle$. Case on the candidate rules for establishing *3)*.

**(SEQ-E$_{\Rightarrow}$)** : Then $T_m = \bullet$.
By (IH), with $\sigma$ as evidence that $C.x_1 := e_1; \ldots; C.x_m := e_m$ can fail, $\bigsqcup_{p=1}^{m-1} \ell_p \sqcup pc \sqsubseteq \ell_m$.
Since $\ell_m \sqsubseteq \ell'$, *4)* holds.

**(SEQ-OK$_{\Rightarrow}$)** : Then $T_m = \texttt{skip}$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_m$. By Lemma 7.4, $\vdash_{\mathsf{dep}} \sigma_m$ and $\Gamma_m \models_{\mathsf{dep}} \sigma_m$.
By Lemma 7.5, $\vdash_{\mathsf{err}} \sigma_m$.
Let $\langle \sigma_m, e_{m+1}\rangle \Rightarrow \langle \sigma', V\rangle$.

Now, $R = \bullet \iff V = \bullet$.
Assume $V = \bullet$.
Then, by (IH), with $\sigma_m$ as evidence that $e_{m+1}$; can fail, $\bigsqcup_{p=1}^{m} \ell_{m+1} \sqcup pc \sqsubseteq \ell_m$.
Since $\ell_{m+1} \sqsubseteq \ell'$, *4)* holds.

□

*Proof of s:* Follows from Lemma 7.6 for $a$, since no semantic rule for statement evaluation introduces an error. To give an impression of how the proof for the various cases goes, we prove one sample case below, namely that of (TRY$_\vdash$).

Assume Lemma 7.6 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.6 holds for $s$ with typing derivation height $n+1$.

**Inductive step:** One (interesting) case to consider.

**(TRY$_\vdash$)** : Then $s = \texttt{try}\ s_1\ \texttt{catch}\ s_2$, for some $s_1$ and $s_2$.
By *1)*, $pc \vdash \Gamma\,\{s_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma \odot \Gamma_1\,\{s_2\}\,\Gamma_2 : \ell_2$,
where $\ell' = \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
By *3)*, $\langle \sigma, s\rangle \Rightarrow \langle \sigma', \bullet\rangle$ for some $\sigma$ and $\sigma'$ for which $\vdash_{\mathsf{err}} \sigma$, $\vdash_{\mathsf{dep}} \sigma$ and $\Gamma \models_{\mathsf{dep}} \sigma$. (*)
Let $\langle \sigma, s_1\rangle \Rightarrow \langle \sigma_1, T_1\rangle$ and $\langle \sigma_1, s_2\rangle \Rightarrow \langle \sigma', T_2\rangle$.
$T_1 = T_2 = \bullet$, else (*) is contradicted.
By Lemma 7.5, $\vdash_{\mathsf{err}} \sigma_1$.
By Lemma 7.4, $\vdash_{\mathsf{dep}} \sigma_1$.
Since $\sigma \sqsubseteq \sigma_1$, $\Gamma \models_{\mathsf{dep}} \sigma_1$.
Since $\Gamma \sqsubseteq \Gamma_1$, $(\Gamma \odot \Gamma_1)^{\mathsf{s}} = \Gamma^{\mathsf{s}}$, and thus $\Gamma \odot \Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
So $\Gamma \odot \Gamma_1 \models_{\mathsf{err}} s_2$, evidenced by $\sigma_1$.
By (IH), $pc \sqcup \ell_1 \sqsubseteq \ell_2$. So $pc \sqsubseteq \ell_2$.
So *4)* holds.

□

**Lemma A.4.** *$\sim_l^\Gamma$ is an equivalence relation.*
*Proof: Recall that $\sigma_1 \sim_l^\Gamma \sigma_2$ iff, for all $C$,*

$$\sigma_1(C) \neq \sigma_2(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell. \qquad (\sim_l^\Gamma)$$

*An equivalence relation is* i) *reflexive,* ii) *symmetric, and* iii) *transitive. We prove that $\sim_l^\Gamma$ has each of these properties now.*

*i) We must show that $\sigma \sim_l^\Gamma \sigma$, for all $\sigma$. This trivially follows from antireflexivity of $\neq$; $\sigma(C) \neq \sigma(C)$, for all $C$ and $\sigma$, meaning $(\sim_l^\Gamma)$ is vacuously true.*

*ii) We must show that $\sigma_1 \sim_l^\Gamma \sigma_2 \implies \sigma_2 \sim_l^\Gamma \sigma_1$, for all $\sigma_i$. Assume $\sigma_1 \sim_l^\Gamma \sigma_2$. Then, for any $C$,*

$$\sigma_1(C) \neq \sigma_2(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell.$$

*By symmetry of $\neq$,*

$$\sigma_1(C) \neq \sigma_2(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell.$$

*As $C$ was arbitrary, we are done.*

*iii) We must show that $\sigma_1 \sim_l^\Gamma \sigma_2 \wedge \sigma_2 \sim_l^\Gamma \sigma_3 \implies \sigma_1 \sim_l^\Gamma \sigma_3$. Assume $\sigma_1 \sim_l^\Gamma \sigma_2$ and $\sigma_2 \sim_l^\Gamma \sigma_3$ hold. We must show that then $\sigma_1 \sim_l^\Gamma \sigma_3$ must hold. For any $C$, we have*

$$\sigma_1(C) \neq \sigma_2(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell \qquad (\sim_{\ell\ 12}^\Gamma)$$
$$\sigma_2(C) \neq \sigma_3(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell. \qquad (\sim_{\ell\ 23}^\Gamma)$$

*We must show that*

$$\sigma_1(C) \neq \sigma_3(C) \wedge \Gamma \models_{\mathsf{err}} \tau(C) \implies \Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell. \qquad (\sim_{\ell\ 13}^\Gamma)$$

*If $\Gamma \models_{\mathsf{err}} \tau(C)$ is false, then $(\sim_{\ell\ 12}^\Gamma)$, $(\sim_{\ell\ 23}^\Gamma)$ and $(\sim_{\ell\ 13}^\Gamma)$ are all vacuously true. Assume $\Gamma \models_{\mathsf{err}} \tau(C)$ is true. If $\sigma_1(C) \neq \sigma_2(C)$ resp. $\sigma_2(C) \neq \sigma_3(C)$, then by $(\sim_{\ell\ 12}^\Gamma)$ resp. $(\sim_{\ell\ 23}^\Gamma)$, $\Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell$, and thus $(\sim_{\ell\ 13}^\Gamma)$ holds. Assume $\sigma_1(C) = \sigma_2(C)$ and $\sigma_2(C) = \sigma_3(C)$. Then $\sigma_1(C) = \sigma_3(C)$ by transitivity of $=$, so $(\sim_{\ell\ 13}^\Gamma)$ is vacuously true.*

□

Raising the $\Gamma^{\mathsf{c}}$ and $\Gamma^{\mathsf{e}}$ makes it more likely for $\sigma_1 \sim_\ell^\Gamma \sigma_2$ to hold, as the conclusion of Definition 7.5 is more likely to hold.

**Lemma A.5** ($\sim_l^\Gamma$ monotone wrt. $\Gamma$)**.** *If $\sigma_1 \sim_l^\Gamma \sigma_2$, $\forall C\,.\,\Gamma^{\mathsf{s}}(C) = \mathtt{B} = \iff \Gamma^{\mathsf{s}}(C) = \mathtt{B}$ and $\Gamma \sqsubseteq \Gamma'$, then $\sigma_1 \sim_l^{\Gamma'} \sigma_2$.*

## Noninterference

**Lemma A.6.** *$=_\ell$ is an equivalence relation.*

*Proof: An equivalence relation is* i) *reflexive,* ii) *symmetric, and* iii) *transitive. We prove that $\sim_\ell$ has each of these properties now.*

*i) We must show that $\sigma =_\ell \sigma$, for all $\sigma$. Parts 1 and 3 of Definition 4.1 hold by reflexivity of $=$, and 2 by idempotence of $\cap$ and $\cup$.*

*ii) We must show that $\sigma_1 =_\ell \sigma_2 \implies \sigma_2 =_\ell \sigma_1$, for all $\sigma_i$. Assume $\sigma_1 =_\ell \sigma_2$. We must show that $\sigma_2 =_\ell \sigma_1$. Parts 1 and 3 of Definition 4.1 hold by symmetry of $=$, and 2 by commutativity of $\cap$ and $\cup$.*

*iii) We must show that $\sigma_1 =_\ell \sigma_2 \wedge \sigma_2 =_\ell \sigma_3 \implies \sigma_1 =_\ell \sigma_3$, for all $\sigma_i$. Assume $\sigma_1 =_\ell \sigma_2$ and $\sigma_2 =_\ell \sigma_3$. We must show that $\sigma_1 =_\ell \sigma_3$. We do this by showing that each of the three parts of Definition 4.1 hold.*

1. *Follows from transitivity of $=$.*
2. *Let $\mathsf{lvl}(C) \sqsubseteq \ell$. Since $\sigma_1 =_\ell \sigma_2$, $\sigma_1(C) = \sigma_2(C)$. Since $\sigma_2 =_\ell \sigma_3$, $\sigma_3(C) = \sigma_2(C)$. By transitivity of $=$, $\sigma_1(C) = \sigma_3(C)$.*
3. *Let $\mathsf{lvl}(C.x) \sqsubseteq \ell$. Then $\mathsf{lvl}(C) \sqsubseteq \ell$. So, by the proof of point 2 above, all $\sigma_i$ agree on $C$. Two cases to consider.*
   - *$C \notin \mathrm{I}(\sigma_i) \cup \mathrm{B}(\sigma_i)$: Vacuously true.*
   - *$C \in \mathrm{I}(\sigma_i) \cup \mathrm{B}(\sigma_i)$: As $\sigma_1 =_\ell \sigma_2$ and $\sigma_2 =_\ell \sigma_3$, $\sigma_1(C.x) = \sigma_2(C.x)$ and $\sigma_2(C.x) = \sigma_3(C.x)$. By transitivity of $=$, $\sigma_1(C.x) = \sigma_3(C.x)$.*

   *As $C$ and $C.x$ were arbitrary, $\sigma_1 =_\ell \sigma_3$*

□

The following lemma follows from the definition of $\odot$.

**Lemma A.7.** *For all $\Gamma_1$ and $\Gamma_2$, $(\Gamma_1 \odot \Gamma_2)^{\mathsf{s}} \sqsubseteq \Gamma_j^{\mathsf{s}}$ and $\Gamma_j^{\mathsf{e}} \sqsubseteq (\Gamma_1 \odot \Gamma_2)^{\mathsf{e}}$.*

**Lemma 7.7** *For all $t$, $\sigma$, $\sigma'$, $\Gamma$, $\Gamma'$, $\ell'$, $pc$, if*

*i) $pc \vdash \Gamma\,\{t\}\,\Gamma' : \ell'$*
*ii) $\vdash_{\mathsf{dep}} \sigma$, $\vdash_{\mathsf{dep}} \Gamma$, $\Gamma \models_{\mathsf{dep}} \sigma$, $\vdash_{\mathsf{err}} \sigma$*
*iii) $\langle \sigma, t\rangle \Rightarrow \langle \sigma', R\rangle$*
*iv) $pc \not\sqsubseteq \ell$,*

*then*

*v) $\sigma \sim_\ell^{\Gamma'} \sigma'$.*
*vi) $\sigma =_\ell \sigma'$.*

*Proof of "$\implies$ v)" for a:* By mutual induction in the height $j$ of the typing derivation of each of $e$, $c$ and $i$.

**Base $e$:** Two cases to consider.

**(NUM$_\vdash$)** : Then $e = n$ for some $n$.
By (NUM$_\Rightarrow$), $\sigma' = \sigma$, so $\sigma \sim_\ell^{\Gamma'} \sigma'$ follows by reflexivity of $\sim_\ell^{\Gamma'}$.

**(VAR$_\vdash$)** : Same argument as in (NUM$_\vdash$) case.

**Base $c$:** Two cases to consider.

**(INIT-T$_\vdash$)** : Then $c = C\,\{i\} = \tau(C)$ for some $C$.
By (INIT-A$_\Rightarrow$), $\sigma' = \sigma$, so $\sigma \sim_\ell^{\Gamma'} \sigma'$ follows by reflexivity of $\sim_\ell^{\Gamma'}$.

**(INIT-S-T$_\vdash$)** : Same argument as in (INIT-T$_\vdash$) case.

**Base $i$:** No case to consider.

We now assume Lemma 7.7 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.7 holds for $e$, $c$ and $i$ with typing derivation height $n+1$.

**Inductive step $e$:** Three cases to consider.

**(OP-T$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some total operator $\oplus$.
Let $\langle \sigma, e_1\rangle \Rightarrow \langle \sigma_1, V_1\rangle$.
By *i)* we have that $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell' = \ell_1 \sqcup \ell_2$.
By (IH), $\sigma \sim_\ell^{\Gamma_1} \sigma_1$.
By Lemmas A.5 and 7.2, $\sigma \sim_\ell^{\Gamma'} \sigma_1$.
Case on the rule used to establish *iii)*.

**(OP-EL$_\Rightarrow$):** Then $\sigma' = \sigma_1$.
So $\sigma \sim_\ell \sigma'$.

**(OP-ER$_\Rightarrow$):** Then $V_1 \neq \bullet$.
By Lemma 7.4, $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
Let $\langle \sigma_1, e_2\rangle \Rightarrow \langle \sigma', V_2\rangle$.

By (IH), $\sigma_1 \sim_\ell^{\Gamma'} \sigma'$.
By transitivity of $\sim_\ell^{\Gamma'}$, $\sigma \sim_\ell^{\Gamma'} \sigma'$.

**(OP-OK$_\Rightarrow$):** Same argument as prior case.

**(OP-P$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some partial operator $\oplus$.
Let $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, V_1 \rangle$.
By *i)* we have that $pc \vdash \Gamma \{e_1\} \Gamma_1 : \ell_1$ and $pc \vdash \Gamma_1 \{e_2\} \Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell = \ell_1 \sqcup \ell_2$.
By (IH), $\sigma \sim_\ell \sigma_1$.
Case on the rule used to establish *iii)*.
All cases and proofs thereof are the same as for (OP-T$_\vdash$), except that (OP-P$_\vdash$) has an extra case:

**(OP-EP$_\Rightarrow$):** Same argument as case (OP-OK$_\Rightarrow$).

**(FIELD$_\vdash$)** : Then $e = C.x$ for some field $C.x$.
We have $pc \vdash \Gamma \{\tau(C)\} \Gamma' : \ell$ by *i)*.
Let $\langle \sigma, \tau(C) \rangle \Rightarrow \langle \sigma', I \rangle$.
By (IH), $\sigma \sim_\ell^{\Gamma'} \sigma'$, regardless of whether (FIELD-E$_\Rightarrow$) or (FIELD-OK$_\Rightarrow$) was used to establish *iii)*.

**Inductive step** $c$**:** Three cases to consider.

**(INIT-F$_\vdash$)** : Then $c = C\,\{i\}$ for some $C$ and $i$. Also, $\Gamma(C) = \text{U}$.
By *i)* we have $pc \sqcup \Gamma^{\mathsf{c}}(C) \sqcup \Gamma^{\mathsf{e}}(C) \vdash \Gamma[C \mapsto^{\mathsf{s}} \text{B}]\,\{i\}\,\Gamma''[C \mapsto^{\mathsf{s}} \text{B}] : \ell_C$ where $\Gamma' = \Gamma''[C \mapsto \langle \text{I}, pc, \ell_C \rangle]$ and $\ell' = \ell_C \sqcup \Gamma''^{\mathsf{e}}(C)$.
Case on the rule used to establish *iii)*.

**(INIT-A$_\Rightarrow$)** : Then $\sigma' = \sigma$, and thus $\sigma \sim_\ell^{\Gamma'} \sigma'$ follows from reflexivity of $\sim_\ell^{\Gamma'}$.

**(INIT-U$_\Rightarrow$)** : Then $\sigma(C) = \text{U}$. Let $\langle \sigma[C \mapsto \text{B}], i \rangle \Rightarrow \langle \sigma''[C \mapsto \text{B}], T \rangle$.
So $\sigma' = \sigma''[C \mapsto I(T)]$.
From *ii)* and $\Gamma(C) = \sigma(C) = \text{U}$, we get $\Gamma[C \mapsto^{\mathsf{s}} \text{B}] \models_{\mathsf{dep}} \sigma[C \mapsto \text{B}]$. Also, from *ii)*, $\vdash_{\mathsf{dep}} \Gamma[C \mapsto^{\mathsf{s}} \text{B}]$ and $\vdash_{\mathsf{dep}} \sigma[C \mapsto \text{B}]$. Also, from *ii)*, $\vdash_{\mathsf{err}} \sigma[C \mapsto \text{B}]$.
By (IH), *ii)* $\sigma[C \mapsto \text{B}] \sim_\ell^{\Gamma''[C \mapsto^{\mathsf{s}} \text{B}]} \sigma''[C \mapsto \text{B}]$.
Since $\sigma(C) = \text{U}$ and $\sigma'(C) \neq \text{U}$, $\Gamma'^{\mathsf{c}}(C) \not\sqsubseteq \ell$ must hold. This follows from the definition of $\Gamma'$ and $pc \not\sqsubseteq \ell$.
If $\sigma'(C) = \bullet$, then furthermore $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$ must hold.
$\sigma'(C) = \bullet$ when $T = \bullet$. By Lemma 7.6, with $\sigma[C \mapsto \text{B}]$ as evidence that $i$ can fail, we get that $pc \sqcup \Gamma^{\mathsf{c}}(C) \sqcup \Gamma^{\mathsf{e}}(C) \sqsubseteq \ell_C$. So $pc \sqsubseteq \ell_C$. $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$ how follows from the definition of $\Gamma'$.

**(INIT-S-FT$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**(INIT-S-FF$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**Inductive step** $i$**:** By (INIT-F$_\vdash$) (the only rule that starts a $i$-typing),
and since $\Gamma \models_{\mathsf{dep}} \sigma$, $\sigma(C) = \text{B}$.
One case to consider.

**(INIT$_\vdash$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$. Induction in $k$.

**Base:** Here, $k = 0$. Then $i = \texttt{skip}$.
By (INIT$_\vdash$), $\Gamma' = \Gamma$. Only (SKIP$_\Rightarrow$) can conclude *iii)*.
(SKIP$_\Rightarrow$) gives $\sigma' = \sigma$. So *v)* holds by reflexivity of $\sim_\ell^{\Gamma'}$.

**Inductive step:** Assume Lemma 7.7 holds for $k \leq m$. This is our induction hypothesis (IH)$_k$. We must show that Lemma 7.7 holds for $k = m + 1$.
Let $\langle \sigma, C.x_1 := e_1; \ldots; C.x_m := e_m \rangle \Rightarrow \langle \sigma_m, T_m \rangle$.
By assumption *i)*, we get that $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1} \{e_q\} \Gamma_q : \ell_q$ for all $q$ from 1 to $m + 1$. $\Gamma = \Gamma_0$ and $\Gamma' = \Gamma_{m+1}$.
By (IH)$_k$, $\sigma \sim_\ell^{\Gamma_m} \sigma_m$.
By Lemma A.5, $\sigma \sim_\ell^{\Gamma'} \sigma_m$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_m$.
By Lemma 7.4, $\Gamma_m \models_{\mathsf{dep}} \sigma_m$ and $\vdash_{\mathsf{dep}} \sigma_m$.
Case on the rule used to establish *iii)*.

**(SEQ-E$_\Rightarrow$)** : Then $\sigma' = \sigma_m$, so *v)* holds.

**(SEQ-OK$_\Rightarrow$)** : We have $\langle \sigma_m, C.x_{m+1} := e_{m+1} \rangle \Rightarrow \langle \sigma', T \rangle$ (*).
Let $\langle \sigma_m, e_{m+1} \rangle \Rightarrow \langle \sigma_e, V_e \rangle$.
By $(IH)$, $\sigma_m \sim_\ell^{\Gamma_{m+1}} \sigma_e$.
Case on $V_e$.

$V_e = \bullet$**:** Then (E-E$_\Rightarrow$) was used to establish (*), so $T = \bullet$ and $\sigma' = \sigma_e$.
Since $\Gamma' = \Gamma_{m+1}$ and $\sigma_m \sim_\ell^{\Gamma_{m+1}} \sigma_e$, *v)* holds by transitivity of $\sim_\ell^{\Gamma'}$.

$V_e = n_e$**:** By Lemma 7.4, $\vdash_{\mathsf{dep}} \sigma_e$ and $\Gamma_{m+1} \models_{\mathsf{dep}} \sigma_e$.
We let $\langle \sigma_e, C.x_{m+1} \rangle \Rightarrow \langle \sigma_{C.x}, V_{C.x} \rangle$ (**).
Since *i)*, $\Gamma_i^{\mathsf{s}}(C) = \text{B}$.
Since $\Gamma_{m+1} \models_{\mathsf{dep}} \sigma_e$, $\sigma_e(C) = \text{B}$.
So the only rule which can conclude (**) is (FIELD-OK$_\Rightarrow$), which will only be able to use (INIT-A$_\Rightarrow$) or (INIT-S-A$_\Rightarrow$). In either case, $\sigma_{C.x} = \sigma_e$ and $V_{C.x} = n_{C.x}$ for some $n_{C.x}$.
Since $\sigma_m \sim_\ell^{\Gamma_{m+1}} \sigma_e$, $\sigma_m \sim_\ell^{\Gamma_{m+1}} \sigma_{C.x}$.
Since $\sigma' = \sigma_{C.x}[C.x \mapsto n_e]$, and $\sigma'$ does not differ from $\sigma_{C.x}$ in initialization statuses, we get by transitivity of $\sim_\ell^{\Gamma_{m+1}}$ that $\sigma \sim_\ell^{\Gamma_{m+1}} \sigma'$.
Since $\Gamma' = \Gamma_{m+1}$, *v)* follows.

□

*Proof of "* $\Longrightarrow$ *v)" for s:* Follows from Lemma 7.7 for $a$, since no semantic rule for statement evaluation introduces an error. To give an impression of how the proof for the various cases goes, we prove one sample case below, namely that of (TRY$_\vdash$).

Assume Lemma 7.7 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.7 holds for $s$ with typing derivation height $n+1$.

**Inductive step:** One (interesting) case to consider.

**(**TRY$_\vdash$**):** Then $s = \texttt{try}\ s_1\ \texttt{catch}\ s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma\,\{s_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma \odot \Gamma_1\,\{s_2\}\,\Gamma_2 : \ell_2$, where $\ell' = \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\sigma \sim_\ell^{\Gamma_1} \sigma_1$.
By Lemmas A.5 and A.7, $\sigma \sim_\ell^{\Gamma \odot \Gamma_1} \sigma_1$ and $\sigma \sim_\ell^{\Gamma_1 \odot \Gamma_2} \sigma_1$.
Case on $T_1$.
$T_1 = \texttt{skip}$**:** Then $\sigma' = \sigma_1$.
So *v)* holds.
$T_1 = \bullet$**:** Let $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By (IH), $\sigma_1 \sim_\ell^{\Gamma_2} \sigma'$.
By Lemmas A.5 and A.7, $\sigma_1 \sim_\ell^{\Gamma_1 \odot \Gamma_2} \sigma'$.
By transitivity of $\sim_\ell^{\Gamma_1 \odot \Gamma_2}$, $\sigma_1 \sim_\ell^{\Gamma_1 \odot \Gamma_2} \sigma'$.
So *v)* holds.

□

*Proof of "* $\Longrightarrow$ *vi)" for a:* By mutual induction in the height $j$ of the typing derivation of each of $e$, $c$ and $i$.

**Base** $e$**:** Two cases to consider.

**(**NUM$_\vdash$**)** : Then $e = n$ for some $n$.
By the sole applicable rule (NUM$_\Rightarrow$), $\sigma' = \sigma$, so $\sigma =_\ell \sigma'$ follows by reflexivity of $=_\ell$.
**(**VAR$_\vdash$**)** : Similar argument as in (NUM$_\vdash$) case.

**Base** $c$**:** Two cases to consider.

**(**INIT-T$_\vdash$**)** : Then $c = C\,\{i\} = \tau(C)$ for some $C$.
By the sole applicable rule (INIT-A$_\Rightarrow$), $\sigma' = \sigma$, so $\sigma =_\ell \sigma'$ follows by reflexivity of $=_\ell$.
**(**INIT-S-T$_\vdash$**)** : Similar argument as in (INIT-T$_\vdash$) case.

**Base** $i$**:** No case to consider.

We now assume Lemma 7.7 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.7 holds for $e$, $c$ and $i$ with typing derivation height $n+1$.

**Inductive step** $e$**:** Three cases to consider.

**(**OP-T$_\vdash$**)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some total operator $\oplus$.
Let $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, V_1 \rangle$.
By *i)* we have that $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_2 \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell = \ell_1 \sqcup \ell_2$.
By (IH), $\sigma =_\ell \sigma_1$. Case on the rule used to establish *iii)*.
**(**OP-EL$_\Rightarrow$**):** Then $\sigma' = \sigma_1$. So $\sigma =_\ell \sigma'$.
**(**OP-ER$_\Rightarrow$**):** Then $V_1 \neq \bullet$.
By Lemmas 7.3 and 7.4, $\vdash_{\mathsf{dep}} \Gamma_1$, $\vdash_{\mathsf{dep}} \sigma_1$ and $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
We have $\langle \sigma_1, e_2 \rangle \Rightarrow \langle \sigma', V_2 \rangle$.
By (IH), $\sigma_1 =_\ell \sigma'$.
By transitivity of $=_\ell$, $\sigma =_\ell \sigma'$.
**(**OP-OK$_\Rightarrow$**):** Same argument as prior case.

**(OP-P$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some partial operator $\oplus$.
Let $\langle \sigma, e_1 \rangle \Rightarrow \langle \sigma_1, V_1 \rangle$.
By *i)* we have that $pc \vdash \Gamma \{e_1\} \Gamma_1 : \ell_1$ and $pc \vdash \Gamma_1 \{e_2\} \Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell = \ell_1 \sqcup \ell_2$.
By (IH), $\sigma =_\ell \sigma_1$.
Case on the rule used to establish *iii)*.
All cases and proofs thereof are the same as for (OP-T$_\vdash$), except that (OP-P$_\vdash$) has an extra case:
**(OP-EP$_\Rightarrow$):** Same argument as case (OP-OK$_\Rightarrow$).

**(FIELD$_\vdash$)** : Then $e = C.x$ for some field $C.x$.
We have $pc \vdash \Gamma \{\tau(C)\} \Gamma' : \ell$ by *i)*.
Let $\langle \sigma, \tau(C) \rangle \Rightarrow \langle \sigma', I \rangle$.
By (IH), $\sigma =_\ell \sigma'$, regardless of whether (FIELD-E$_\Rightarrow$) or (FIELD-OK$_\Rightarrow$) was used to establish *iii)*.

**Inductive step** $c$**:** Three cases to consider.

**(INIT-F$_\vdash$)** : Then $c = C\ \{i\} = \tau(C)$ for some $C$.
Also, $\Gamma(C) = \mathrm{U}$.
By *i)* we have $pc \sqcup \Gamma^{\mathsf{c}}(C) \sqcup \Gamma^{\mathsf{e}}(S) \vdash \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \{i\} \Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}] : \ell_C$ where $\Gamma' = \Gamma''[C \mapsto \langle \mathrm{I}, pc, \ell_C \rangle]$ and $\ell' = \ell_C \sqcup \Gamma''^{\mathsf{e}}(C)$
Case on the rule used to establish *iii)*.
**(INIT-A$_\Rightarrow$)** : Then $\sigma' = \sigma$, and thus $\sigma =_\ell \sigma'$ follows from reflexivity of $=_\ell$.
**(INIT-U$_\Rightarrow$)** : Let $\langle \sigma[C \mapsto \mathrm{B}], i \rangle \Rightarrow \langle \sigma''[C \mapsto \mathrm{B}], T \rangle$.
So $\sigma' = \sigma''[C \mapsto I(T)]$.
From *ii)* and $\Gamma(C) = \sigma(C) = \mathrm{U}$,
we get $\Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \models_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$.
Also, from *ii)*, $\vdash_{\mathsf{dep}} \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]$ and $\vdash_{\mathsf{dep}} \sigma[C \mapsto \mathrm{B}]$.
$\sigma[C \mapsto \mathrm{B}] =_\ell \sigma''[C \mapsto \mathrm{B}]$ follows from (IH).
By definition of $=_\ell$ and $\sigma'$, $\sigma =_\ell \sigma'$, regardless of the value of $T$.
**(INIT-S-FT$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.
**(INIT-S-FF$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**Inductive step** $i$**:** One case to consider.

**(INIT$_\vdash$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$.
Induction in $k$.

**Base:** Here, $k = 0$. Then $i = \texttt{skip}$.
By (INIT$_\vdash$), $\Gamma' = \Gamma$.
Only (SKIP$_\Rightarrow$) can conclude *iii)*.
(SKIP$_\Rightarrow$) gives $\sigma' = \sigma$.
So *vi)* holds by reflexivity of $=_\ell$.

**Inductive step:** Assume Lemma 7.7 holds for $k \leq m$.
This is our induction hypothesis (IH)$_k$.
We must show that Lemma 7.7 holds for $k = m + 1$.
Let $\langle \sigma, C.x_1 := e_1; \ldots; C.x_m := e_m \rangle \Rightarrow \langle \sigma_m, T_m \rangle$.
By assumption *i)*, we get that $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1} \{e_q\} \Gamma_q : \ell_q$ for all $q$ from 1 to $m + 1$. $\Gamma' = \Gamma_{m+1}$.
By (IH)$_k$, $\sigma =_\ell \sigma_m$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_m$.
By Lemma 7.4, $\Gamma_m \models_{\mathsf{dep}} \sigma_m$ and $\vdash_{\mathsf{dep}} \sigma_m$.
Case on the rule used to establish *iii)*.
**(SEQ-E$_\Rightarrow$)** : Then $\sigma' = \sigma_m$, so *vi)* holds.
**(SEQ-OK$_\Rightarrow$)** : We have $\langle \sigma_m, C.x_{m+1} := e_{m+1} \rangle \Rightarrow \langle \sigma', T \rangle$ (*).
Let $\langle \sigma_m, e_{m+1} \rangle \Rightarrow \langle \sigma_e, V_e \rangle$.
By $(IH)$, $\sigma_m =_\ell \sigma_e$, so $\sigma =_\ell \sigma_e$ by transitivity of $=_\ell$.
Case on $V_e$.
$V_e = \bullet$**:** Then (E-E$_\Rightarrow$) was used to establish (*).
So $T = \bullet$ and $\sigma' = \sigma_e$.
Since $\sigma_m =_\ell \sigma_e$, *vi)* holds by transitivity of $=_\ell$.
$V_e = n_e$**:** By Lemma 7.4, $\vdash_{\mathsf{dep}} \sigma_e$ and $\Gamma_{m+1} \models_{\mathsf{dep}} \sigma_e$.
We let $\langle \sigma_e, C.x_{m+1} \rangle \Rightarrow \langle \sigma_{C.x}, V_{C.x} \rangle$ (**).
Since *i)*, $\Gamma_j^{\mathsf{s}}(C) = \mathrm{B}$.
Since $\Gamma_{m+1} \models_{\mathsf{dep}} \sigma_e$, $\sigma_e(C) = \mathrm{B}$.

So the only rule which can conclude (**) is (FIELD-OK$_\Rightarrow$), which will only be able to use (INIT-A$_\Rightarrow$) or (INIT-S-A$_\Rightarrow$).
In either case, $\sigma_{C.x} = \sigma_e$ and $V_{C.x} = n_{C.x}$ for some $n_{C.x}$.
Since $\sigma =_\ell \sigma_e$, $\sigma =_\ell \sigma_{C.x}$ by transitivity of $=_\ell$.
Since $\sigma' = \sigma_{C.x}[C.x \mapsto n_e]$, and $pc \not\sqsubseteq \ell$, we get from *i)* that $\sigma_{C.x} =_\ell \sigma_{C.x}[C.x \mapsto n_e]$.
By transitivity of $=_\ell$, $\sigma =_\ell \sigma'$. So *vi)* holds.

□

*Proof of "$\implies$ vi)" for s:* By induction in the height $j$ of the typing derivation of $s$.

**Base:** Three cases to consider.

**(SKIP$_\vdash$):** Then $s = \texttt{skip}$.
*iii)* is impossible, so *vi)* is vacuously true.

**(VAR-A$_\vdash$):** Then $s = x := e$ for some $x$ and $e$.
By *i)*, $pc \vdash \Gamma\,\{e\}\,\Gamma' : \ell'$ and $pc \sqsubseteq \mathsf{lvl}(x)$. (*)
(VAR-A$_\Rightarrow$) and (E-E$_\Rightarrow$) can establish *iii)*.
In both cases, $e$ is evaluated under $\sigma$.
Let $\langle \sigma, e\rangle \Rightarrow \langle \sigma'', V\rangle$.
By Lemma 7.7 for $a$, $\sigma =_\ell \sigma''$.
Case on $V$.

$V = \bullet$**:** Then *iii)* was established through (E-E$_\Rightarrow$).
By (E-E$_\Rightarrow$), $\sigma' = \sigma''$.
So *vi)* holds.

$V = n$**:** Then *iii)* was established through (VAR-A$_\Rightarrow$).
By (VAR-A$_\Rightarrow$), $\sigma' = \sigma''[x \mapsto n]$.
By (*), $\sigma'' =_\ell \sigma'$.
By transitivity, $\sigma =_\ell \sigma'$.
So *vi)* holds.

**(FIELD-A$_\vdash$):** Then $s = C.x := e$ for some $C.x$ and $e$.
By *i)*, $pc \vdash \Gamma\,\{e\}\,\Gamma'' : \ell_e$, $pc \sqcup \ell_e \vdash \Gamma''\,\{C.x\}\,\Gamma' : \ell_{C.x}$, $\ell' = \ell_e \sqcup \ell_{C.x}$ and $pc \sqsubseteq \mathsf{lvl}(x)$. (*)
(FIELD-A-E$_\Rightarrow$), (FIELD-A-OK$_\Rightarrow$) and (E-E$_\Rightarrow$) can establish *iii)*.
In all cases, $e$ is evaluated under $\sigma$.
Let $\langle \sigma, e\rangle \Rightarrow \langle \sigma_e, V_e\rangle$.
By Lemma 7.7 for $a$, $\sigma =_\ell \sigma_e$.
Case on $V_e$.

$V_e = \bullet$**:** Then *iii)* was established through (E-E$_\Rightarrow$).
By (E-E$_\Rightarrow$), $\sigma' = \sigma''$.
So *vi)* holds.

$V_e = n$**:** Then *iii)* was established through either (FIELD-A-E$_\Rightarrow$) or (FIELD-A-OK$_\Rightarrow$).
In both cases, $C.x$ is evaluated under $\sigma_e$.
Let $\langle \sigma_e, C.x\rangle \Rightarrow \langle \sigma_{C.x}, V_{C.x}\rangle$.
By Lemmas 7.3, 7.4 and 7.5, $\vdash_{\mathsf{dep}} \Gamma'' \vdash_{\mathsf{dep}} \sigma_e$, $\Gamma'' \models_{\mathsf{dep}} \sigma_e$ and $\vdash_{\mathsf{err}} \sigma_e$.
By Lemma 7.7 for $a$, $\sigma_e =_\ell \sigma_{C.x}$.
By transitivity, $\sigma =_\ell \sigma_{C.x}$.
Case on $V_{C.x}$.

$V_{C.x} = \bullet$**:** Then *iii)* was established through (FIELD-A-E$_\Rightarrow$).
By (FIELD-A-E$_\Rightarrow$), $\sigma' = \sigma_{C.x}$.
So *vi)* holds.

$V_{C.x} \neq \bullet$**:** Then *iii)* was established through (FIELD-A-OK$_\Rightarrow$).
By (FIELD-A-OK$_\Rightarrow$), $\sigma' = \sigma_{C.x}[C.x \mapsto n]$.
By (*), $\sigma_{C.x} =_\ell \sigma'$.
By transitivity, $\sigma =_\ell \sigma'$.
So *vi)* holds.

Assume Lemma 7.7 holds for $s$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.7 holds for $s$ with typing derivation height $n + 1$.

**Inductive step:** Four cases to consider.

**(SEQ$_\vdash$):** Then $s = s_1; s_2$ for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma\,\{s_1\}\,\Gamma_1 : \ell_1$, $pc \sqcup \ell_1 \vdash \Gamma_1\,\{s_2\}\,\Gamma' : \ell_2$ and $\ell' = \ell_1 \sqcup \ell_2$.

(SEQ-E$_{\Rightarrow}$) and (SEQ-OK$_{\Rightarrow}$) can establish *iii)*.
In both cases, $s_1$ is evaluated under $\sigma$.
Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\sigma =_\ell \sigma_1$.
Case on $T_1$.

$T_1 = \bullet$**:** Then *iii)* was established through (SEQ-E$_{\Rightarrow}$).
By (SEQ-E$_{\Rightarrow}$), $\sigma' = \sigma_1$.
So *vi)* holds.

$T_1 = \texttt{skip}$**:** Then *iii)* was established through (SEQ-OK$_{\Rightarrow}$).
By (SEQ-E$_{\Rightarrow}$), $s_2$ is evaluated under $\sigma_1$.
Let $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By Lemmas 7.3, 7.4 and 7.5, $\vdash_{\mathsf{dep}} \Gamma_1$ $\vdash_{\mathsf{dep}} \sigma_1$, $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$ and $\vdash_{\mathsf{err}} \sigma_1$.
By (IH), $\sigma_1 =_\ell \sigma'$.
By transitivity, $\sigma =_\ell \sigma'$.
So *vi)* holds.

**(IF$_{\vdash}$):** Then $s = \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2$ for some $e$, $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma \{e\} \Gamma_e : \ell_e$, $pc \sqcup \ell_e \vdash \Gamma_e \{s_1\} \Gamma_1 : \ell_1$, and $pc \sqcup \ell_e \vdash \Gamma_e \{s_2\} \Gamma_2 : \ell_2$, where $\Gamma' = \Gamma_1 \odot \Gamma_2$ and $\ell' = \ell_1 \sqcup \ell_2$.
(IF-T$_{\Rightarrow}$), (IF-F$_{\Rightarrow}$) and (E-E$_{\Rightarrow}$) can establish *iii)*.
In both cases, $e$ is evaluated under $\sigma$.
Let $\langle \sigma, e \rangle \Rightarrow \langle \sigma_e, V_e \rangle$.
By Lemma 7.7 for $a$, $\sigma =_\ell \sigma_e$.
Case on $V_e$.

$V_e = \bullet$**:** Then *iii)* was established through (E-E$_{\Rightarrow}$).
By (E-E$_{\Rightarrow}$), $\sigma' = \sigma_e$.
So *vi)* holds.

$V_e = \bar{0}$**:** Then *iii)* was established through (IF-T$_{\Rightarrow}$).
By (IF-T$_{\Rightarrow}$), $s_1$ is evaluated under $\sigma_e$.
Let $\langle \sigma_e, s_1 \rangle \Rightarrow \langle \sigma', R \rangle$.
By Lemmas 7.3, 7.4 and 7.5, $\vdash_{\mathsf{dep}} \Gamma_e$ $\vdash_{\mathsf{dep}} \sigma_e$, $\Gamma_e \models_{\mathsf{dep}} \sigma_e$ and $\vdash_{\mathsf{err}} \sigma_e$.
By (IH), $\sigma_e =_\ell \sigma'$.
By transitivity, $\sigma =_\ell \sigma'$.
So *vi)* holds.

$V_e = 0$**:** Near-identical argument as in the $V_e = \bar{0}$ case.

**(TRY$_{\vdash}$):** Then $s = \texttt{try}\ s_1\ \texttt{catch}\ s_2$, for some $s_1$ and $s_2$.
By *i)*, $pc \vdash \Gamma \{s_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma \odot \Gamma_1 \{s_2\} \Gamma_2 : \ell_2$,
where $\ell' = \ell_2$ and $\Gamma' = \Gamma_1 \odot \Gamma_2$.
(TRY-E$_{\Rightarrow}$) and (TRY-OK$_{\Rightarrow}$) can establish *iii)*.
In both cases, $s_1$ is evaluated under $\sigma$.
Let $\langle \sigma, s_1 \rangle \Rightarrow \langle \sigma_1, T_1 \rangle$.
By (IH), $\sigma =_\ell \sigma_1$.
Case on $T_1$.

$T_1 = \texttt{skip}$**:** Then *iii)* was established through (TRY-OK$_{\Rightarrow}$).
By (TRY-OK$_{\Rightarrow}$), $\sigma' = \sigma_1$.
So *vi)* holds.

$T_1 = \bullet$**:** Then *iii)* was established through (TRY-E$_{\Rightarrow}$).
By (TRY-E$_{\Rightarrow}$), $s_2$ is evaluated under $\sigma_1$.
Let $\langle \sigma_1, s_2 \rangle \Rightarrow \langle \sigma', R \rangle$.
By Lemmas 7.3, 7.4 and 7.5, $\vdash_{\mathsf{dep}} \Gamma_1$ $\vdash_{\mathsf{dep}} \sigma_1$, $\Gamma_1 \models_{\mathsf{dep}} \sigma_1$ and $\vdash_{\mathsf{err}} \sigma_1$.
Since $\Gamma \sqsubseteq \Gamma_1$, $(\Gamma \odot \Gamma_1)^{\mathsf{s}} = \Gamma^{\mathsf{s}}$.
From this, and Lemma A.2, $\vdash_{\mathsf{dep}} \Gamma \odot \Gamma_1$ and $\Gamma \odot \Gamma_1 \models_{\mathsf{dep}} \sigma_1$.
By (IH), $\sigma_1 =_\ell \sigma'$.
By transitivity, $\sigma =_\ell \sigma'$.
So *vi)* holds.

**(WHILE$_{\vdash}$):** Then $\hat{s} = \texttt{while}\ e\ \texttt{do}\ s$, for some $e$ and $s$.
By *i)*, $pc \sqcup \ell_i \vdash \Gamma_i \{e\} \Gamma'_i : \ell^e_i$ and $pc \sqcup \ell_i \sqcup \ell^e_i \sqcup \mathsf{lvl}(e) \vdash \Gamma'_i \{s\} \Gamma_{i+1} : \ell^s_i$,
where $\ell_0 = \bot$, $\ell_{i+1} = \ell_i \sqcup \ell^e_i \sqcup \ell^s_i$,

$i = 0..n$, $(\Gamma_n, \ell_n) = (\Gamma_{n+1}, \ell_{n+1})$,
$\ell' = \ell_n$, $\Gamma = \Gamma_0$ and $\Gamma' = \bigodot_{j=0}^{n} \Gamma'_j \odot \Gamma_{j+1}$.
From this, and since the type system is deterministic,
we have for all $k > n$ that $(\Gamma_k, \ell_k) = (\Gamma_{k-1}, \ell_{k-1})$.
By transitivity, $(\Gamma_k, \ell_k) = (\Gamma_n, \ell_n)$.
So $pc \sqcup \ell_k \vdash \Gamma_k \{e\} \Gamma'_k : \ell^e_k$ and $pc \sqcup \ell_k \sqcup \ell^e_k \sqcup \mathsf{lv}(e) \vdash \Gamma'_k \{s\} \Gamma_{k+1} : \ell^s_k$,
where $\ell_{k+1} = \ell_k \sqcup \ell^e_k \sqcup \ell^s_k$, and $k > n$.
By Lemma 7.3, for all $j \geq 0$, $\vdash_{\mathsf{dep}} \Gamma'_j$ and $\vdash_{\mathsf{dep}} \Gamma_{j+1}$ (*).
Let $\sigma_0 = \sigma$, $\langle \sigma_j, e \rangle \Rightarrow \langle \sigma^e_j, V_j \rangle$, $\langle \sigma^e_j, s \rangle \Rightarrow \langle \sigma^s_j, T_j \rangle$ and $\sigma_{j+1} = \sigma^s_j$.
We have that, for some $j$,
$\sigma'$ equals either $\sigma^e_j$ ($e$ evaluates to $0$ or $\bullet$) or $\sigma^s_j$ ($s$ evaluates to $\bullet$).
This follows mainly from the observation that

a) if $V_j \in \{0, \bullet\}$,
   $\langle \sigma_j, \texttt{while } e \texttt{ do } s \rangle \Rightarrow \langle \sigma^e_j, \hat{T}_j \rangle$,
   where $\hat{T} = \bullet$ if $V_j = \bullet$ and $\hat{T}_j = \texttt{skip}$ if $V_j = 0$,
b) otherwise, if $T_j = \bullet$,
   $\langle \sigma_j, \texttt{while } e \texttt{ do } s \rangle \Rightarrow \langle \sigma^s_j, T_j \rangle$,
   and
c) otherwise,
   $\langle \sigma_j, \texttt{while } e \texttt{ do } s \rangle \Rightarrow \langle \hat{\sigma}, \hat{T} \rangle$
   where $\hat{\sigma}$ and $\hat{T}$ are defined by
   $\langle \sigma^s_j, \texttt{while } e \texttt{ do } s \rangle \Rightarrow \langle \hat{\sigma}, \hat{T} \rangle$.

which follows from (E-E$_\Rightarrow$), (WHILE-F$_\Rightarrow$), (WHILE-T$_\Rightarrow$), (SEQ-E$_\Rightarrow$) and (SEQ-OK$_\Rightarrow$).
It is therefore sufficient for us to prove that, for all $j$,

1′) $\sigma_j =_\ell \sigma^e_j$, and
2′) $\sigma^e_j =_\ell \sigma^s_j$.

To establish this, we also need, assuming

1) $\vdash_{\mathsf{dep}} \sigma_j$, $\vdash_{\mathsf{err}} \sigma_j$, $\Gamma_j \models_{\mathsf{dep}} \sigma_j$,

to prove

2) $\vdash_{\mathsf{dep}} \sigma^e_j$, $\vdash_{\mathsf{err}} \sigma^e_j$, $\Gamma'_j \models_{\mathsf{dep}} \sigma^e_j$,
3) $\vdash_{\mathsf{dep}} \sigma^s_j$, $\vdash_{\mathsf{err}} \sigma^s_j$, $\Gamma_{j+1} \models_{\mathsf{dep}} \sigma_s$,

We will then get *vi)* by transitivity of $=_\ell$.
Let $j$ be arbitrary.
Assume 1).
By Lemma 7.7 for $a$, 1′) holds.
By (*) and Lemmas 7.4 and 7.5, 2) holds.
By (IH), 2′) holds.
By (*) and Lemmas 7.4 and 7.5, 3) holds.

□

**Lemma A.8.** *For all $e$ and $\sigma$,*
*if $\langle \sigma, e \rangle \Rightarrow \langle \sigma', \_ \rangle$,*
*then for all $\sigma''$, if $\sigma' \sqsubseteq \sigma''$ and $\sigma'(C) = \mathrm{B} \iff \sigma''(C) = \mathrm{B}$ for all $C$, $\langle \sigma'', e \rangle \Rightarrow \langle \sigma'', \_ \rangle$.*

**Lemma 7.8** *For all $t$, $\sigma$, $\sigma'$, $\Gamma$, $\Gamma'$, $\ell$, $pc$, if*

*i)* $pc \vdash \Gamma \{t\} \Gamma' : \ell'$
*ii)* $(\vdash_{\mathsf{dep}} \Gamma)$, $(\vdash_{\mathsf{dep}} \sigma_j)$, $(\Gamma \models_{\mathsf{dep}} \sigma_j)$
*iii)* $(\vdash_{\mathsf{err}} \sigma_j)$
*iv)* $\langle \sigma_j, t \rangle \Rightarrow \langle \sigma'_j, R_j \rangle$
*v)* $\sigma_1 \sim_\ell \sigma_2$, $\sigma_1 =_\ell \sigma_2$

*then*

*vi)* $R_j \neq \bullet = R_{\bar{j}} \implies l' \not\sqsubseteq l$
*vii)* $\sigma_1 \sim_\ell \sigma_2$
*viii)* $\sigma_1 =_\ell \sigma_2$

*Proof for $a$:* By mutual induction in the height $z$ of the typing derivation of each of $e$, $c$ and $i$.

**Base $e$:** Two cases to consider.

**(NUM$_\vdash$)** : Then $e = n$ for some $n$.
By (NUM$_\Rightarrow$), $\sigma'_j = \sigma_j$.
$R_1 \neq \bullet \neq R_2$, so *vi)* holds vacuously.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Now *vii)* and *viii)* follow from $\sigma'_j = \sigma_j$.
**(VAR$_\vdash$)** : Same argument as in (NUM$_\vdash$) case.

**Base** $c$**:** Two cases to consider.

**(INIT-T$_\vdash$)** : Then $c = C\,\{i\} = \tau(C)$ for some $C$.
By (INIT-A$_\Rightarrow$), $\sigma'_j = \sigma_j$.
$R_1 \neq \bullet \neq R_2$, so *vi)* holds vacuously.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Now *vii)* and *viii)* follow from $\sigma'_j = \sigma_j$.
**(INIT-S-T$_\vdash$)** : Same argument as in (INIT-T$_\vdash$) case.

**Base** $i$**:** No case to consider.

We now assume Lemma 7.8 holds for $e$, $c$ and $i$ with typing derivation height $\leq n$. This is our induction hypothesis, (IH). We must show that Lemma 7.8 holds for $e$, $c$ and $i$ with typing derivation height $n+1$.

**Inductive step** $e$**:** Three cases to consider.

**(OP-T$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some total operator $\oplus$.
Let $\langle \sigma_j, e_1 \rangle \Rightarrow \langle \sigma_{1_j}, V_{1_j} \rangle$.
By *i)* we have that $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell = \ell_1 \sqcup \ell_2$.
By (IH), $V_{1_j} \neq \bullet = V_{1_{\bar{j}}} \implies \ell_1 \not\sqsubseteq \ell$, $\sigma_{1_1} \sim_\ell^{\Gamma_1} \sigma_{1_2}$, and $\sigma_{1_1} =_\ell \sigma_{1_2}$.
Three cases to consider for the values of $V_{1_j}$ (all other cases are either symmetric, or have a near-identical argument).

$V_{1_1} = \bullet$, $V_{1_2} = \bullet$**:** *iv)* can only be established through (OP-EL$_\Rightarrow$) for $j = 1$ and $j = 2$ , and by that rule, $R_j = V_{1_j} = \bullet$ and $\sigma'_j = \sigma_{1_j}$.
So *vi)* holds vacuously.
We have from earlier that $\sigma_{1_1} =_\ell \sigma_{1_2}$ and $\sigma_{1_1} \sim_\ell^{\Gamma_1} \sigma_{1_2}$.
By Lemmas A.5 and 7.2, $\sigma_{1_1} \sim_\ell^{\Gamma'} \sigma_{1_2}$.
Together, this gives *vii)* and *viii)*.

$V_{1_1} = \bullet$, $V_{1_2} \neq \bullet$**:** *iv)* can only be established through (OP-EL$_\Rightarrow$) for $j = 1$ , and by that rule, $R_1 = V_{1_1} = \bullet$ and $\sigma'_1 = \sigma_{1_1}$.
By (IH), we have $V_{1_j} \neq \bullet = V_{1_{\bar{j}}} \implies \ell_1 \not\sqsubseteq \ell$. Since $\ell_1 \sqsubseteq \ell'$ and $V_{1_1} = \bullet \neq V_{1_2}$, we get $\ell' \not\sqsubseteq \ell$, so *vi)* holds.
Let $\langle \sigma_{1_2}, e_2 \rangle \Rightarrow \langle \sigma'_2, V_{2_2} \rangle$.
This is established either through (OP-ER$_\Rightarrow$) or (OP-EP$_\Rightarrow$).
Regardless of which, since $pc \sqcup \ell_1 \not\sqsubseteq \ell$, we get from Lemmas 7.7 and 7.7 that $\sigma_{1_2} \sim_\ell^{\Gamma'} \sigma'_2$ and $\sigma_{1_2} =_\ell \sigma'_2$.
By Lemmas A.5 and 7.2, $\sigma_{1_1} \sim_\ell^{\Gamma'} \sigma_{1_2}$.
Together, this gives *vii)* and *viii)*.

$V_{1_1} \neq \bullet$, $V_{1_2} \neq \bullet$**:** Then *iv)* was established by either (OP-ER$_\Rightarrow$) or (OP-OK$_\Rightarrow$).
Regardless of which, $\langle \sigma_{1_j}, e_2 \rangle \Rightarrow \langle \sigma'_j, V_{2_j} \rangle$.
Since $pc \vdash \Gamma\,\{e_1\}\,\Gamma_2 : \ell_1$, Lemmas 7.3 and 7.4 give us $\vdash_{\mathsf{dep}} \Gamma_1$, $\vdash_{\mathsf{dep}} \sigma_{1_j}$ and $\Gamma_1 \models_{\mathsf{dep}} \sigma_{1_j}$.
By (IH), $V_{2_j} \neq \bullet = V_{2_{\bar{j}}} \implies \ell_2 \not\sqsubseteq \ell$, $\sigma'_1 \sim_\ell^{\Gamma'} \sigma'_2$, and $\sigma'_1 =_\ell \sigma'_2$.
So *vii)* and *viii)* hold.
*vi)* holds since $R_j = \bullet \iff V_{2_j} = \bullet$ and $\ell_2 \sqsubseteq \ell'$.

**(OP-P$_\vdash$)** : Then $e = e_1 \oplus e_2$ for some $e_i$ and some partial operator $\oplus$.
Let $\langle \sigma_j, e_1 \rangle \Rightarrow \langle \sigma_{1_j}, V_{1_j} \rangle$.
By *i)* we have that $pc \vdash \Gamma\,\{e_1\}\,\Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1\,\{e_2\}\,\Gamma' : \ell_2$ for some $\Gamma_1$ and $\ell_i$ where $\ell = \ell_1 \sqcup \ell_2$.
By (IH), $V_{1_j} \neq \bullet = V_{1_{\bar{j}}} \implies \ell_1 \not\sqsubseteq \ell$, $\sigma_{1_1} \sim_\ell^{\Gamma_1} \sigma_{1_2}$, and $\sigma_{1_1} =_\ell \sigma_{1_2}$.
Three cases to consider for the values of $V_{1_j}$ (all other cases are either symmetric, or have a near-identical argument).
All cases and proofs thereof are the same as for (OP-T$_\vdash$), with the following addition:

$V_{1_1} \neq \bullet$, $V_{1_2} \neq \bullet$**:** as in (OP-T$_\vdash$), except *i)* could also have been established through (OP-EP$_\Rightarrow$).
*vii)* and *vii)* are established through the same argument as in (OP-T$_\vdash$).
For *vi)*, observe that $\mathsf{lvl}(e_1) \sqcup \mathsf{lvl}(e_2) \sqsubseteq \ell'$.
If $R_1 \neq R_2$, then either $V_{1_1} \neq V_{1_2}$ or $V_{2_1} \neq V_{2_2}$ (or both).
Since $\sigma_1 =_\ell \sigma_2$ and $\sigma_{1_1} =_\ell \sigma_{1_2}$, this difference can only occur if $e_1$ or $e_2$ has a variable or field with security level $\hat{\ell}$ where $\hat{\ell} \not\sqsubseteq \ell$.

But in that case, $\mathsf{lvl}(e_1) \sqcup \mathsf{lvl}(e_2) \not\sqsubseteq \ell$. Thus $\ell' \not\sqsubseteq \ell$.

**(FIELD$_\vdash$)** : Then $e = C.x$ for some $C.x$.
We have $pc \vdash \Gamma\,\{\tau(C)\}\,\Gamma' : \ell'$ by *i)*.
Let $\langle \sigma_j, \tau(C)\rangle \Rightarrow \langle \sigma'_j, I_j\rangle$.
By (IH), $I_j \neq \bullet = I_{\bar{j}} \implies \ell' \not\sqsubseteq \ell$, $\sigma'_1 \sim_\ell^{\Gamma'} \sigma'_2$ and $\sigma'_1 =_\ell \sigma'_2$.
So *vii)* and *viii)* hold. As $R_j = \bullet \iff I_j = \bullet$, *vi)* follows.

**Inductive step** $c$**:** Three cases to consider.

**(INIT-F$_\vdash$)** : Then $c = C\,\{i\} = \tau(C)$ for some $C$.
Also, $\Gamma(C) = \mathrm{U}$.
By *i)* we have $pc \sqcup \Gamma^{\mathsf{c}}(C) \sqcup \Gamma^{\mathsf{e}}(C) \vdash \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]\,\{i\}\,\Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}] : \ell_C$ where $\Gamma' = \Gamma''[C \mapsto \langle \mathrm{I}, pc, \ell_C\rangle]$ and $\ell' = \ell_C \sqcup \Gamma'^{\mathsf{e}}(C)$.
Case on $\sigma_j(C)$.

$\sigma_1(C) \neq \mathrm{U}$, $\sigma_2(C) \neq \mathrm{U}$**:** Then (INIT-A$_\Rightarrow$) was used to establish *iv)*, for $j = 1$ and $j = 2$.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Since $\sigma'_j = \sigma_j$, *vii)* and *viii)* hold.
For *vi)* we must consider $\sigma_j(C)$.
From *ii)* and $\Gamma^{\mathsf{s}}(C) = \mathrm{U}$, $\sigma_j(C) \neq \mathrm{B}$.
If $\sigma_1(C) = \sigma_2(C)$, $R_1 = R_2$, so *vi)* holds vacuously.
If $\sigma_1(C) \neq \sigma_2(C)$, then $\sigma_1(C) = \mathrm{I}$, $\sigma_2(C) = \bullet$ (or vice versa).
Then since $\sigma'_1 \sim_\ell^{\Gamma'} \sigma'_2$, by Definition 7.5 pt. **??**, $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$.
By definition of $\ell'$, *vi)* holds.

$\sigma_1(C) = \mathrm{U}$, $\sigma_2(C) = \mathrm{U}$**:** Then (INIT-U$_\Rightarrow$) was used to establish *iv)*, for $j = 1$ and $j = 2$.
From *v)*, $\sigma_1[C \mapsto \mathrm{B}] \sim_\ell^{\Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]} \sigma_2[C \mapsto \mathrm{B}]$
and $\sigma_1[C \mapsto \mathrm{B}] =_\ell \sigma_2[C \mapsto \mathrm{B}]$.
By *ii)*, $\vdash_{\mathsf{dep}} \Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}]$, $\vdash_{\mathsf{dep}} \sigma_j[C \mapsto \mathrm{B}]$
and $\Gamma[C \mapsto^{\mathsf{s}} \mathrm{B}] \models_{\mathsf{dep}} \sigma_j[C \mapsto \mathrm{B}]$.
By (INIT-U$_\Rightarrow$), $\langle \sigma_j[C \mapsto \mathrm{B}], i\rangle \Rightarrow \langle \sigma''_j[C \mapsto \mathrm{B}], T_j\rangle$,
with $\sigma'_j = \sigma''_j[C \mapsto I(T_j)]$.
By (IH), we get that $T_j \neq \bullet = T_{\bar{j}} \implies \ell_C \not\sqsubseteq \ell$,
$\sigma''_1[C \mapsto \mathrm{B}] \sim_\ell^{\Gamma''[C \mapsto^{\mathsf{s}} \mathrm{B}]} \sigma''_2[C \mapsto \mathrm{B}]$, and $\sigma''_1[C \mapsto \mathrm{B}] =_\ell \sigma''_2[C \mapsto \mathrm{B}]$.
This immediately gives us *vi)*, since $R_j = \bullet \iff T_j = \bullet$ and $\ell_C \sqsubseteq \ell'$.
By further observing that $\sigma'_j(C) \neq \mathrm{U}$ and $\ell_C \sqsubseteq \Gamma'^{\mathsf{e}}(C)$, we get *vii)*.
By (INIT$_\vdash$) (used to type $i$), $\ell_C \sqsubseteq C.x$, for all fields $C.x$ of $C$.
So, even if $\sigma'_1(C) \neq \sigma'_2(C)$, *viii)* still holds since then, $\ell_C \not\sqsubseteq \ell$.

$\sigma_1(C) = \mathrm{U}$, $\sigma_2(C) \neq \mathrm{U}$**:** Then (INIT-U$_\Rightarrow$), resp. (INIT-A$_\Rightarrow$), was used to establish *iv)* for $j = 1$, resp. $j = 2$ (or vice versa).
By (INIT-U$_\Rightarrow$) we have $\langle \sigma_1[C \mapsto \mathrm{B}], i\rangle \Rightarrow \langle \sigma''_1[C \mapsto \mathrm{B}], T_1\rangle$, with $\sigma'_1 = \sigma''_1[C \mapsto I(T_1)]$ and $R_1 = \bullet \iff T_1 = \bullet$.
By (INIT-A$_\Rightarrow$), $\sigma'_2 = \sigma_2$ and $R_2 = I(\sigma_2(C))$.
By $\sigma_1 \sim_\ell^{\Gamma} \sigma_2$, $\Gamma^{\mathsf{c}}(C) \not\sqsubseteq \ell$.
By Lemmas 7.7, Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma'_1$.
By Lemma 7.7, $\sigma_1 =_\ell \sigma'_1$.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
For *vii)* and *viii)*, by transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$, it remains only to show that $\sigma_2 \sim_\ell^{\Gamma'} \sigma'_2$ and $\sigma_2 =_\ell \sigma'_2$.
This follows from $\sigma'_2 = \sigma_2$.
Since $\Gamma^{\mathsf{s}}(C) = \mathrm{U}$, then by *ii)*, $\sigma_2(C) \neq \mathrm{B}$.
By (INIT-U$_\Rightarrow$), $\sigma'_1(C) \neq \mathrm{B}$.
If $\sigma'_1(C) = \sigma'_2(C)$, *vi)* holds vacuously.
If $\sigma'_1(C) \neq \sigma'_2(C)$, then either $\sigma'_1(C) = \bullet$ or $\sigma'_2(C) = \bullet$ (not both).
In the latter case, $\Gamma^{\mathsf{e}}(C) \not\sqsubseteq \ell$, so $\sigma_1 \sim_\ell^{\Gamma} \sigma_2$ with Lemmas A.5 and 7.2 gives $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$.
In the former case, $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$ follows directly from *vii)*.
So either way, $\Gamma'^{\mathsf{e}}(C) \not\sqsubseteq \ell$.
Since $\Gamma'^{\mathsf{e}}(C) \sqsubseteq \ell'$, *vi)* holds.

**(INIT-S-FT$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**(INIT-S-FF$_\vdash$)** : Argument similar in style as in (INIT-F$_\vdash$) case.

**Inductive step** $i$**:** One case to consider.

**(INIT$_\vdash$)** : Then $i = C.x_1 := e_1; \ldots; C.x_k := e_k$. Induction in $k$.

**Base:** Here, $k = 0$. Then $i = \texttt{skip}$.
By (INIT$_\vdash$), $\Gamma' = \Gamma$.
Only (SKIP$_\Rightarrow$) can conclude *iv)*.
(SKIP$_\Rightarrow$) gives $\sigma'_j = \sigma_j$.
So *vii)* and *viii)* holds by reflexivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$.
*vi)* holds vacuously as $R_j = \texttt{skip}$.

**Inductive step:** Assume Lemma 7.8 holds for $k \leq m$.
This is our induction hypothesis (IH)$_k$.
We must show that Lemma 7.8 holds for $k = m + 1$.
Let $\langle \sigma_j, C.x_1 := e_1; \ldots; C.x_m := e_m \rangle \Rightarrow \langle \sigma_{m_j}, T_{m_j} \rangle$ (*)$_j$.
By assumption *i)*, we get that $\bigsqcup_{p=1}^{q-1} \ell_p \sqcup pc \vdash \Gamma_{q-1} \{e_q\} \Gamma_q : \ell_q$ for all $q$ from 1 to $m+1$. $\Gamma' = \Gamma_{m+1}$.
By (IH)$_k$, $R_j \neq \bullet = R_{\vec{j}} \implies \bigsqcup_{p=1}^{m-1} \ell_p \not\sqsubseteq \ell$, $\sigma_{m_1} \sim_\ell^{\Gamma_m} \sigma_{m_2}$ and $\sigma_{m_1} =_\ell \sigma_{m_2}$.
By Lemmas A.5 and 7.2, $\sigma_{m_1} \sim_\ell^{\Gamma'} \sigma_{m_2}$.
By Lemma 7.3, $\vdash_{\mathsf{dep}} \Gamma_m$.
By Lemma 7.4, $\Gamma_m \models_{\mathsf{dep}} \sigma_m$ and $\vdash_{\mathsf{dep}} \sigma_m$.
By Lemma 7.5, $\vdash_{\mathsf{err}} \sigma_{m_j}$.
Three cases to consider for the values of $T_{m_j}$ (all other cases are either symmetric, or have a near-identical argument).

$T_{m_1} = \bullet$, $T_{m_2} = \bullet$: Then *iv)*, for $j = 1$ and $j = 2$, were both established by (SEQ-E$_\Rightarrow$).
So $\sigma'_j = \sigma_{m_j}$.
Thus *vii)* and *vii)* follow from reflexivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.
Since $R_1 = R_2 = \bullet$, *vi)* holds vacuously.

$T_{m_1} \neq \bullet$, $T_{m_2} = \bullet$: Then $\bigsqcup_{p=1}^{m-1} \ell_p \not\sqsubseteq \ell$, so $\bigsqcup_{p=1}^{m-1} \ell_p \sqcup pc \not\sqsubseteq \ell$.
By Lemma 7.6, $\ell_m \not\sqsubseteq \ell$.
Since $\ell_m \sqsubseteq \ell'$, $\ell' \not\sqsubseteq \ell$, so *vi)* holds.
*iv)*, for $j = 1$ resp. $j = 2$, was established by (SEQ-OK$_\Rightarrow$) resp. (SEQ-E$_\Rightarrow$).
So $\sigma'_2 = \sigma_{m_2}$, and thus by transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$,
$\sigma_{m_1} \sim_\ell^{\Gamma'} \sigma'_2$ and $\sigma_{m_1} =_\ell \sigma'_2$.
To show *vii)* and *viii)*, by transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$, it remains only to be shown that $\sigma'_1 \sim_\ell^{\Gamma'} \sigma_{m_1}$ and $\sigma'_1 =_\ell \sigma_{m_1}$.
Let $\langle \sigma_{m_1}, e_{m+1} \rangle \Rightarrow \langle \sigma_{e_1}, V_{e_1} \rangle$.
By Lemmas 7.7 and 7.7, $\sigma_{m_1} \sim_\ell^{\Gamma'} \sigma_{e_1}$ and $\sigma_{m_1} =_\ell \sigma_{e_1}$.
Let $\langle \sigma_{e_1}, C.x_{m+1} \rangle \Rightarrow \langle \sigma_{C.x_1}, V_{C.x_1} \rangle$ (*).
By *i)*, $\Gamma_p^{\mathsf{s}}(C) = \text{B}$.
Since $\Gamma_m \models_{\mathsf{dep}} \sigma_{e_1}$, $\sigma_{e_1}(C) = \text{B}$.
So the only rule which can conclude (*) is (FIELD-OK$_\Rightarrow$), which will only be able to use (INIT-A$_\Rightarrow$) or (INIT-S-A$_\Rightarrow$).
In either case, $\sigma_{C.x_1} = \sigma_{e_1}$ and $V_{C.x} = n_{C.x}$ for some $n_{C.x}$.
So $\sigma_{m_1} =_\ell \sigma_{C.x_1}$ and $\sigma_{m_1} \sim_\ell^{\Gamma'} \sigma_{C.x_1}$.
Since $\sigma'_1 = \sigma_{C.x_1}[C.x \mapsto n_e]$, and $\bigsqcup_{p=1}^{m-1} \ell_p \sqcup pc \not\sqsubseteq \ell$, we get from *i)* that $\sigma_{m_1} =_\ell \sigma'_1$.
Since $\sigma'_1$ and $\sigma_{C.x_1}$ do not differ in initialization statuses, $\sigma_{m_1} \sim_\ell \sigma'_1$.
So *vii)* and *viii)* holds by transitivity of $=_\ell$ and $\sigma =_\ell \sigma'$.

$T_{m_1} \neq \bullet$, $T_{m_2} \neq \bullet$: Then *iv)*, for $j = 1$ and $j = 2$, were both established by (SEQ-OK$_\Rightarrow$).
Let $\langle \sigma_{m_j}, e_{m+1} \rangle \Rightarrow \langle \sigma_{e_j}, V_{e_j} \rangle$.
By (IH), $V_{e_j} \neq \bullet = V_{e_{\vec{j}}} \implies \ell_{m+1} \not\sqsubseteq \ell$,
$\sigma_{m_j} \sim_\ell^{\Gamma'} \sigma_{e_j}$ and $\sigma_{m_j} =_\ell \sigma_{e_j}$.
Since $\ell_{m+1} \sqsubseteq \ell'$, *vi)* holds.
Three cases to consider for the values of $V_{e_j}$ (all other cases are either symmetric, or have a near-identical argument).

$V_{e_1} = \bullet$, $V_{e_2} = \bullet$: Then $\sigma'_j = \sigma_{e_j}$, so *vii)* and *viii)* follow by transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.
$V_{e_1} \neq \bullet$, $V_{e_2} = \bullet$: Then $\sigma'_1 = \sigma_{e_1}$.
Let $\langle \sigma_{e_2}, C.x_{m+1} \rangle \Rightarrow \langle \sigma_{C.x_2}, V_{C.x_2} \rangle$ (*).
By *i)*, $\Gamma_p^{\mathsf{s}}(C) = \text{B}$.
Since $\Gamma_m \models_{\mathsf{dep}} \sigma_{e_2}$, $\sigma_{e_2}(C) = \text{B}$.
So the only rule which can conclude (*) is (FIELD-OK$_\Rightarrow$), which will only be able to use (INIT-A$_\Rightarrow$) or (INIT-S-A$_\Rightarrow$).

In either case, $\sigma_{C.x_2} = \sigma_{e_2}$ and $V_{C.x_2} = n_{C.x_2}$
for some $n_{C.x_2}$.
So $\sigma_{e_2} =_\ell \sigma_{C.x_2}$ and $\sigma_{e_2} \sim_\ell^{\Gamma'} \sigma_{C.x_2}$.
So *vii)* and *viii)* follow from transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.

$V_{e_1} \neq \bullet$, $V_{e_2} \neq \bullet$**:** Let $\langle \sigma_{e_j}, C.x_{m+1} \rangle \Rightarrow \langle \sigma_{C.x_j}, V_{C.x_j} \rangle$ (*).
By *i)*, $\Gamma_p^{\mathsf{s}}(C) = \text{B}$.
Since $\Gamma_m \models_{\mathsf{dep}} \sigma_{e_j}$, $\sigma_{e_j}(C) = \text{B}$.
So the only rule which can conclude (*) is (FIELD-OK$_\Rightarrow$), which will only be able to use (INIT-A$_\Rightarrow$) or (INIT-S-A$_\Rightarrow$).
In either case, $\sigma_{C.x_j} = \sigma_{e_j}$ and $V_{C.x_j} = n_{C.x_j}$
for some $n_{C.x_j}$.
So $\sigma_{e_j} =_\ell \sigma_{C.x_j}$ and $\sigma_{e_j} \sim_\ell^{\Gamma'} \sigma_{C.x_j}$.
So *vii)* and *viii)* follow from transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.

□

*Proof for s:* By induction in the typing of $s$.

**Base:** Three cases to consider.

**(**SKIP$_\vdash$**)** : Then $s = \texttt{skip}$ and $\Gamma' = \Gamma$.
Only (SKIP$_\Rightarrow$) can conclude iv).
(SKIP$_\Rightarrow$) gives $\sigma_j' = \sigma_j$.
So vii) and viii) holds by reflexivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$.
vi) holds vacuously as $R_j = \texttt{skip}$.

**(**VAR-A$_\vdash$**)** : Then $s = x := e$ for some $x$ and $e$.
Also, $pc \vdash \Gamma \{e\} \Gamma' : \ell'$.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Two rules can conclude iv); Only (VAR-A$_\Rightarrow$) and (E-E$_\Rightarrow$).
Regardless of which is used, $e$ is evaluated.
Let $\langle \sigma_j, e \rangle \Rightarrow \langle \sigma_{e_j}, V_{e_j} \rangle$.
By Lemma 7.8, $V_{e_j} \neq \bullet = V_{e_{\bar{j}}} \implies \ell' \not\sqsubseteq \ell$, $\sigma_{e_1} =_\ell \sigma_{e_2}$ and $\sigma_{e_1} \sim_\ell^{\Gamma'} \sigma_{e_2}$.
Since $R_j = \bullet \iff V_{e_j} = \bullet$, vi) holds.
Case on $V_{e_k}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).

$V_{e_1} \neq \bullet \neq V_{e_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (VAR-A$_\Rightarrow$).
By this rule, $\sigma_j' = \sigma_{e_j}[x \mapsto V_{e_j}]$.
By definition of $\sim_\ell^{\Gamma'}$, $\sigma_{e_j} \sim_\ell^{\Gamma'} \sigma_j'$, so by transitivity, vii) holds.
Case on $\mathsf{lvl}(x)$.
$\mathsf{lvl}(x) \not\sqsubseteq \ell$**:** Then by definition of $=_\ell$, $\sigma_{e_j} =_\ell \sigma_j'$, so by transitivity, viii) holds.
$\mathsf{lvl}(x) \sqsubseteq \ell$**:** Then since $e$ is well-typed, $\mathsf{lvl}(e), pc, \ell' \sqsubseteq \mathsf{lvl}(x)$.
So $\mathsf{lvl}(e) \sqsubseteq \mathsf{lvl}(x)$, and therefore, $V_{e_1} = V_{e_2}$.
Thus by definition of $=_\ell$, $\sigma_{e_j} =_\ell \sigma_j'$, so by transitivity, viii) holds.

$V_{e_1} = \bullet = V_{e_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (E-E$_\Rightarrow$).
By this rule, $\sigma_j' = \sigma_{e_j}$.
So by reflexivity and transitivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$, vii) and viii) hold.

$V_{e_1} \neq \bullet = V_{e_2}$**:** Then iv) for $j = 1$, resp. $j = 2$, was established through (VAR-A$_\Rightarrow$), resp. (E-E$_\Rightarrow$).
So, $\sigma_1' = \sigma_{e_1}$ and $\sigma_2' = \sigma_{e_2}[x \mapsto V_{e_2}]$.
By definition of $\sim_\ell^{\Gamma'}$ (no class initialization status difference), $\sigma_{e_j} \sim_\ell^{\Gamma'} \sigma_j'$, so by transitivity, vii) holds.
Since $V_{e_1} \neq \bullet = V_{e_2}$, $\ell' \not\sqsubseteq \ell$.
So vi) holds.
By i), $\mathsf{lvl}(e), pc, \ell' \sqsubseteq \mathsf{lvl}(x)$.
So $\mathsf{lvl}(x) \not\sqsubseteq \ell$.
Thus by definition of $=_\ell$, $\sigma_{e_j} =_\ell \sigma_j'$, so by transitivity, viii) holds.

**(**FIELD-A$_\vdash$**)** : Then $s = C.x := e$ for some $C$, $x$ and $e$.
Also, $pc \vdash \Gamma \{e\} \Gamma_e : \ell_e$ and $pc \vdash \Gamma_e \{e\} \Gamma' : \ell_C$ for some $\Gamma_e$, $\ell_e$ and $\ell_C$ such that $\ell' = \ell_e \sqcup \ell_C$.
By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma_e} \sigma_2$ and $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Three rules can conclude iv); (E-E$_\Rightarrow$), (FIELD-A-OK$_\Rightarrow$), and (FIELD-A-E$_\Rightarrow$).
Regardless of which is used, $e$ is evaluated.
Let $\langle \sigma_j, e \rangle \Rightarrow \langle \sigma_{e_j}, V_{e_j} \rangle$.
By Lemma 7.8, $V_{e_j} \neq \bullet = V_{e_{\bar{j}}} \implies \ell_e \not\sqsubseteq \ell$, $\sigma_{e_1} =_\ell \sigma_{e_2}$ and $\sigma_{e_1} \sim_\ell^{\Gamma_e} \sigma_{e_2}$.

By definition of $\ell'$, $V_{e_j} \neq \bullet = V_{e_{\bar{j}}} \implies \ell' \not\sqsubseteq \ell$ holds.
Case on $V_{e_k}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).
$V_{e_1} = \bullet = V_{e_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (E-E$_{\Rightarrow}$).
By this rule, $\sigma'_j = \sigma_{e_j}$.
So by reflexivity and transitivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$, vii) and viii) hold.
Also, by (E-E$_{\Rightarrow}$), $R_j = \bullet$.
So vi) holds vacuously since $R_1 = \bullet = R_2$.
$V_{e_1} \neq \bullet \neq V_{e_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through either (FIELD-OK$_{\Rightarrow}$) or (FIELD-E$_{\Rightarrow}$).
Regardless of which, $C.x$ is evaluated.
Let $\langle \sigma_{e_j}, C.x \rangle \Rightarrow \langle \sigma_{C.x_j}, V_{C.x_j} \rangle$.
By Lemma 7.8, $V_{C.x_j} \neq \bullet = V_{C.x_{\bar{j}}} \implies \ell_{C.x} \not\sqsubseteq \ell$, $\sigma_{C.x_1} =_\ell \sigma_{C.x_2}$ and $\sigma_{C.x_1} \sim_\ell^{\Gamma'} \sigma_{C.x_2}$.
By definition of $\ell'$, $V_{C.x_j} \neq \bullet = V_{C.x_{\bar{j}}} \implies \ell' \not\sqsubseteq \ell$ holds.
Case on $V_{C.x_k}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).
$V_{C.x_1} = \bullet = V_{C.x_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (FIELD-A-E$_{\Rightarrow}$).
By this rule, $\sigma'_j = \sigma_{C.x_j}$.
So by reflexivity and transitivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$, vii) and viii) hold.
Also, by (FIELD-A-E$_{\Rightarrow}$), $R_j = \bullet$.
So vi) holds vacuously since $R_1 = \bullet = R_2$.
$V_{C.x_1} \neq \bullet \neq V_{C.x_2}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (FIELD-OK$_{\Rightarrow}$).
By this rule, $\sigma'_j = \sigma_{C.x_j}[C.x \mapsto V_{e_j}]$.
By definition of $\sim_\ell^{\Gamma'}$, $\sigma_{C.x_j} \sim_\ell^{\Gamma'} \sigma'_j$, so by transitivity, vii) holds.
Case on $\mathsf{lvl}(C.x)$.
$\mathsf{lvl}(C.x) \not\sqsubseteq \ell$**:** Then by definition of $=_\ell$, $\sigma_{C.x_j} =_\ell \sigma'_j$, so by transitivity, viii) holds.
$\mathsf{lvl}(C.x) \sqsubseteq \ell$**:** By i), $\mathsf{lvl}(e), pc \sqcup \ell_e, \ell' \sqsubseteq \mathsf{lvl}(C.x)$.
So $\mathsf{lvl}(e) \sqsubseteq \mathsf{lvl}(C.x)$, and therefore, $V_{e_1} = V_{e_2}$.
Thus by definition of $=_\ell$, $\sigma_{e_j} =_\ell \sigma'_j$, so by transitivity, viii) holds.
Also, by (FIELD-A-E$_{\Rightarrow}$), $R_j = \texttt{skip}$.
So vi) holds vacuously since $R_1 \neq \bullet \neq R_2$.
$V_{C.x_1} \neq \bullet = V_{C.x_2}$**:** Then iv), for $j = 1$, resp. $j = 2$, was established through (FIELD-A-OK$_{\Rightarrow}$), resp. (FIELD-A-E$_{\Rightarrow}$).
So, $\sigma'_1 = \sigma_{C.x_1}$ and $\sigma'_2 = \sigma_{C.x_2}[C.x \mapsto V_{e_2}]$.
By definition of $\sim_\ell^{\Gamma'}$ (no class initialization status difference), $\sigma_{C.x_j} \sim_\ell^{\Gamma'} \sigma'_j$, so by transitivity, vii) holds.
Since $V_{C.x_1} \neq \bullet = V_{C.x_2}$, $\ell_C \not\sqsubseteq \ell$.
So by definition of $\ell'$, $\ell' \not\sqsubseteq \ell$, so vi) holds.
By i), $\mathsf{lvl}(e), pc \sqcup \ell_e, \ell' \sqsubseteq \mathsf{lvl}(C.x)$.
Since $\ell_C \not\sqsubseteq \ell$, $\mathsf{lvl}(C.x) \not\sqsubseteq \ell$.
Thus by definition of $=_\ell$, $\sigma_{C.x_j} =_\ell \sigma'_j$, so by transitivity, viii) holds.
$V_{e_1} \neq \bullet = V_{e_2}$**:** Then iv), for $j = 1$, was established through either (FIELD-A-OK$_{\Rightarrow}$) or (FIELD-A-E$_{\Rightarrow}$), and iv), for $j = 2$, was established through (E-E$_{\Rightarrow}$).
So, $\sigma'_2 = \sigma_{e_2}$.
Since $V_{e_1} \neq \bullet = V_{e_2}$, $\ell_e \not\sqsubseteq \ell$.
By definition of $\ell'$, $\ell' \not\sqsubseteq \ell$, so vi) holds.
Regardless of which of (FIELD-A-OK$_{\Rightarrow}$) and (FIELD-A-E$_{\Rightarrow}$) were used to establish iv) for $j = 1$, $C.x$ is evaluated.
Let $\langle \sigma_{e_2}, C.x \rangle \Rightarrow \langle \sigma_{C.x_2}, V_{C.x_2} \rangle$.
Depending on the value of $V_{C.x_2}$, either $\sigma_2 = \sigma_{C.x_2}$ or $\sigma_2 = \sigma_{C.x_2}[C.x \mapsto V_{e_2}]$.
Since $C.x$ is evaluated under context $pc \sqcup \ell_e$, we get by Lemmas 7.7 and 7.7 that $\sigma_{e_2} \sim_\ell^{\Gamma'} \sigma'_2$ and $\sigma_{e_2} =_\ell \sigma'_2$, regardless of which of these two possible instances of $\sigma'_j$ we have.
So, by reflexivity and transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$, vii) and viii) hold.

**Inductive step:** Induction hypothesis:

(IH$_s$)**:** Assume the result holds for all $s'$ structurally smaller than $s$.

We proceed by case on the typing of $s$.

**(**SEQ$_\vdash$**)** : Then $s = s_1; s_2$ for some $s_1$ and $s_2$.
By $i$), $pc \vdash \Gamma \{s_1\} \Gamma_1 : \ell_1$ and $pc \sqcup \ell_1 \vdash \Gamma_1 \{s_2\} \Gamma' : \ell_2$, where $\ell' = \ell_1 \sqcup \ell_2$.

By Lemmas A.5 and 7.2, $\sigma_1 \sim_\ell^{\Gamma_1} \sigma_2$ and $\sigma_1 \sim_\ell^{\Gamma'} \sigma_2$.
Two rules can conclude iv); (SEQ-OK$_\Rightarrow$) and (SEQ-E$_\Rightarrow$).
Regardless of which is used, $s_1$ is evaluated.
Let $\langle \sigma_j, s_1 \rangle \Rightarrow \langle \sigma_{s_{1j}}, T_{s_{1j}} \rangle$.
By (IH)$_s$, $T_{s_{1j}} \neq \bullet = T_{s_{1\bar{j}}} \implies \ell_1 \not\sqsubseteq \ell$, $\sigma_{s_{11}} =_\ell \sigma_{s_{12}}$ and $\sigma_{s_{11}} \sim_\ell^{\Gamma_1} \sigma_{s_{12}}$.
By definition of $\ell'$, $T_{s_{1j}} \neq \bullet = T_{s_{1\bar{j}}} \implies \ell' \not\sqsubseteq \ell$ holds.
By Lemmas 7.3, 7.4 and 7.5, $(\vdash_{\mathsf{dep}} \Gamma_1)$, $(\vdash_{\mathsf{dep}} \sigma_{s_{1j}})$, $(\Gamma_1 \models_{\mathsf{dep}} \sigma_{s_{1j}})$ and $(\vdash_{\mathsf{err}} \sigma_{s_{1j}})$.
Case on $T_{s_{1j}}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).
$T_{s_{11}} = \bullet = T_{s_{12}}$: Then iv), for $j = 1$ and $j = 2$, was established through (SEQ-E$_\Rightarrow$).
So, $\sigma'_j = \sigma_{s_{1j}}$.
Thus vii) and viii) hold.
Since $R_1 = \bullet = R_2$, vi) holds vacuously.
$T_{s_{11}} \neq \bullet \neq T_{s_{12}}$: Then iv), for $j = 1$ and $j = 2$, was established through (SEQ-OK$_\Rightarrow$).
In both runs, $s_2$ is evaluated.
Let $\langle \sigma_{s_{1j}}, s_2 \rangle \Rightarrow \langle \sigma_{s_{2j}}, T_{s_{2j}} \rangle$.
By (SEQ-OK$_\Rightarrow$), $\sigma_j = \sigma_{s_{2j}}$.
By (IH)$_s$, $T_{s_{2j}} \neq \bullet = T_{s_{2\bar{j}}} \implies \ell_2 \not\sqsubseteq \ell$, $\sigma_{s_{21}} =_\ell \sigma_{s_{22}}$ and $\sigma_{s_{21}} \sim_\ell^{\Gamma'} \sigma_{s_{22}}$.
Thus vii) and viii) hold.
By definition of $\ell'$, $T_{s_{1j}} \neq \bullet = T_{s_{1\bar{j}}} \implies \ell' \not\sqsubseteq \ell$ holds.
Since $R_j = T_{s_{2j}}$, vi) holds.
$T_{s_{11}} \neq \bullet = T_{s_{12}}$: Then iv), for $j = 1$, resp. $j = 2$, was established through (SEQ-OK$_\Rightarrow$), resp. (SEQ-E$_\Rightarrow$).
So $\sigma'_2 = \sigma_{s_{12}}$
Since $T_{s_{11}} \neq \bullet = T_{s_{12}}$, $\ell_1 \not\sqsubseteq \ell$, and thus $\ell' \not\sqsubseteq \ell$, holds, so vi) holds.
The $j = 1$ run evaluates $s_2$.
Let $\langle \sigma_{s_{11}}, s_2 \rangle \Rightarrow \langle \sigma_{s_{21}}, T_{s_{21}} \rangle$.
We have $\sigma'_1 = \sigma_{s_{21}}$.
Since $\ell_1 \not\sqsubseteq \ell$ holds, and therefore $pc \sqcup \ell_1 \not\sqsubseteq \ell$, by Lemmas **??** and **??**, $\sigma_{s_{11}} \sim_\ell^{\Gamma'} \sigma'_1$ and $\sigma_{s_{11}} =_\ell \sigma'_1$.
By reflexivity and transitivity of $\sim_\ell^\Gamma$ and $=_\ell$, vii) and viii) hold.

**(IF$_\vdash$)** : Then $s = \texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2$ for some $e$, $s_1$ and $s_2$.
By $i$), $pc \vdash \Gamma \{e\} \Gamma_0 : \ell_e$ and $pc \sqcup \mathsf{lvl}(e) \sqcup \ell_e \vdash \Gamma_0 \{s_k\} \Gamma_k : \ell_k$ for $k = 1$ and $k = 2$, with $\Gamma' = \Gamma_1 \odot \Gamma_2$ and $\ell' = \ell_e \sqcup \ell_1 \sqcup \ell_2$
Three rules can conclude iv), for both $j = 1$ and $j = 2$; (E-E$_\Rightarrow$), (IF-F$_\Rightarrow$) and (IF-T$_\Rightarrow$).
Regardless of which is used, $e$ is evaluated.
Let $\langle \sigma_j, e \rangle \Rightarrow \langle \sigma_{e_j}, V_{e_j} \rangle$.
By Lemma 7.8, $V_{e_j} \neq \bullet = V_{e_{\bar{j}}} \implies \ell_e \not\sqsubseteq \ell$, $\sigma_{e_1} =_\ell \sigma_{e_2}$ and $\sigma_{e_1} \sim_\ell^{\Gamma_0} \sigma_{e_2}$.
By Lemmas A.5 and 7.2, $\sigma_{e_1} \sim_\ell^{\Gamma_1} \sigma_{e_2}$, $\sigma_{e_1} \sim_\ell^{\Gamma_2} \sigma_{e_2}$ and $\sigma_{e_1} \sim_\ell^{\Gamma'} \sigma_{e_2}$ (since $\Gamma_k^{\mathsf{c}} \sqsubseteq \Gamma'^{\mathsf{c}}$ and $\Gamma_k^{\mathsf{e}} \sqsubseteq \Gamma'^{\mathsf{e}}$).
By definition of $\ell'$, $V_{e_j} \neq \bullet = V_{e_{\bar{j}}} \implies \ell' \not\sqsubseteq \ell$ holds.
By Lemmas 7.3, 7.4 and 7.5, $(\vdash_{\mathsf{dep}} \Gamma_0)$, $(\vdash_{\mathsf{dep}} \sigma_{e_j})$, $(\Gamma_0 \models_{\mathsf{dep}} \sigma_{e_j})$ and $(\vdash_{\mathsf{err}} \sigma_{e_j})$.
Case on $V_{e_j}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).
$V_{e_1} = \bullet = V_{e_2}$: Then iv), for both $j = 1$ and $j = 2$, was established through (E-E$_\Rightarrow$).
By this rule, $\sigma'_j = \sigma_{e_j}$.
So by reflexivity and transitivity of $=_\ell$ and $\sim_\ell^{\Gamma'}$, vii) and viii) hold.
Also, by (E-E$_\Rightarrow$), $R_j = \bullet$.
So vi) holds vacuously since $R_1 = \bullet = R_2$.
$V_{e_1} \neq \bullet \neq V_{e_2}$: Then iv), for both $j = 1$ and $j = 2$, was established through either (IF-F$_\Rightarrow$) or (IF-T$_\Rightarrow$).
The rule used depends on the value of $V_{e_j}$. Case on $V_{e_j}$.
$V_{e_1} = 0 = V_{e_2}$ Then iv), for both $j = 1$ and $j = 2$, was established through (IF-F$_\Rightarrow$).
In both runs, $s_2$ is evaluated.
So $\langle \sigma_{e_j}, s_2 \rangle \Rightarrow \langle \sigma'_j, R_j \rangle$.
By (IH)$_s$, $R_j \neq \bullet = R_{\bar{j}} \implies \ell_2 \not\sqsubseteq \ell$, $\sigma'_1 =_\ell \sigma'_2$ and $\sigma'_1 \sim_\ell^{\Gamma'} \sigma'_2$, so vii) and viii) hold.
Since $\ell_2 \sqsubseteq \ell'$, vii) holds.
$V_{e_1} \neq 0 \neq V_{e_2}$ Then iv), for both $j = 1$ and $j = 2$, was established through (IF-T$_\Rightarrow$).
In both runs, $s_1$ is evaluated.
So $\langle \sigma_{e_j}, s_1 \rangle \Rightarrow \langle \sigma'_j, R_j \rangle$.
By (IH)$_s$, $R_j \neq \bullet = R_{\bar{j}} \implies \ell_1 \not\sqsubseteq \ell$, $\sigma'_1 =_\ell \sigma'_2$ and $\sigma'_1 \sim_\ell^{\Gamma'} \sigma'_2$, so vii) and viii) hold.
Since $\ell_1 \sqsubseteq \ell'$, vii) holds.

$V_{e_1} \neq 0 = V_{e_2}$ Then iv), for both $j = 1$, resp. $j = 2$, was established through either (IF-F$_\Rightarrow$) or (IF-T$_\Rightarrow$).
So $\langle \sigma_{e_j}, s_{k_j} \rangle \Rightarrow \langle \sigma'_j, R_j \rangle$, with $k_1 \neq k_2$.
Since $V_{e_1} \neq V_{e_2}$ and $\sigma_{e_1} =_\ell \sigma_{e_2}$, $\mathsf{lvl}(e) \not\sqsubseteq \ell$.
Since $s_1$ and $s_2$ would both be run under context $pc \sqcup \mathsf{lvl}(e) \sqcup \ell_e$, we get from Lemmas 7.7 and 7.7 that $\sigma_{e_j} \sim_\ell^{\Gamma'} \sigma'_j$ and $\sigma_{e_j} =_\ell \sigma'_j$, regardless of which of (IF-F$_\Rightarrow$) and (IF-T$_\Rightarrow$) is used to prove iv) for $j = 1$ and $j = 2$.
So vii) and viii) both hold.
By Lemma 7.6, we get that $pc \sqcup \mathsf{lvl}(e) \sqcup \ell_e \sqsubseteq \ell_k$. So $\ell' \not\sqsubseteq \ell$. Thus vi) holds.

$V_{e_1} \neq \bullet = V_{e_2}$**:** Then iv), for $j = 2$, was established through (E-E$_\Rightarrow$), and iv), for $j = 1$, was established through either (IF-F$_\Rightarrow$) or (IF-T$_\Rightarrow$).
So $\sigma'_2 = \sigma_{e_2}$.
Since $V_{e_1} \neq \bullet = V_{e_2}$, $\ell_e \not\sqsubseteq \ell$ and $\ell' \not\sqsubseteq \ell$, so vi) holds.
Also, $\langle \sigma_{e_1}, s_{k_1} \rangle \Rightarrow \langle \sigma'_1, R_1 \rangle$, with $k_1 \in \{1, 2\}$.
Since $s_1$ and $s_2$ would both be run under context $pc \sqcup \mathsf{lvl}(e) \sqcup \ell_e$, we get from Lemmas 7.7 and 7.7 that $\sigma_{e_1} \sim_\ell^{\Gamma'} \sigma'_1$ and $\sigma_{e_1} =_\ell \sigma'_1$, regardless of which of (IF-F$_\Rightarrow$) and (IF-T$_\Rightarrow$) is used to prove iv) for $j = 1$.
So vii) and viii) both hold by reflexicity and transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.

**(**TRY$_\vdash$**)** : Then $s = \texttt{try}\ s_t\ \texttt{catch}\ s_c$ for some $s_t$ and $s_c$.
By *i)*, $pc \vdash \Gamma\,\{s\}\,\Gamma_t : \ell_t$ and $pc \sqcup \ell_t \vdash \Gamma \odot \Gamma_t\,\{s_c\}\,\Gamma_c : \ell'$, with $\Gamma' = \Gamma_t \odot \Gamma_c$.
Two rules can conclude iv), for both $j = 1$ and $j = 2$; (TRY-E$_\Rightarrow$) and (TRY-OK$_\Rightarrow$).
Regardless of which is used, $s_t$ is evaluated.
Let $\langle \sigma_j, s_t \rangle \Rightarrow \langle \sigma_{s_{t_j}}, T_{s_{t_j}} \rangle$.
By (IH)$_s$, $T_{s_{t_j}} \neq \bullet = T_{s_{t_{\bar{j}}}} \implies \ell_t \not\sqsubseteq \ell$, $\sigma_{s_{t_1}} =_\ell \sigma_{s_{t_2}}$ and $\sigma_{s_{t_1}} \sim_\ell^{\Gamma_t} \sigma_{s_{t_2}}$.
By Lemmas A.5 and 7.2, $\sigma_{s_{t_1}} \sim_\ell^{\Gamma \odot \Gamma_t} \sigma_{s_{t_2}}$, $\sigma_{s_{t_1}} \sim_\ell^{\Gamma_c} \sigma_{s_{t_2}}$ and $\sigma_{s_{t_1}} \sim_\ell^{\Gamma'} \sigma_{s_{t_2}}$ (since $\Gamma_t^{\mathsf{c}} \sqsubseteq (\Gamma \odot \Gamma_t)^{\mathsf{c}}$, $\Gamma_t^{\mathsf{e}} \sqsubseteq (\Gamma \odot \Gamma_t)^{\mathsf{e}}$, $\Gamma_c^{\mathsf{c}} \sqsubseteq \Gamma'^{\mathsf{c}}$ and $\Gamma_t^{\mathsf{e}} \sqsubseteq \Gamma'^{\mathsf{e}}$).
By Lemmas 7.3, 7.4 and 7.5, $(\vdash_{\mathsf{dep}} \Gamma_t)$, $(\vdash_{\mathsf{dep}} \sigma_{s_{t_j}})$, $(\Gamma_t \models_{\mathsf{dep}} \sigma_{s_{t_j}})$ and $(\vdash_{\mathsf{err}} \sigma_{s_{t_j}})$.
Since $(\Gamma \odot \Gamma_t)^{\mathsf{s}} = \Gamma^{\mathsf{s}}$, $(\vdash_{\mathsf{dep}} \Gamma \odot \Gamma_t)$, and $(\Gamma \odot \Gamma_t \models_{\mathsf{dep}} \sigma_{s_{t_j}})$.
Case on $T_{s_{t_j}}$. Three cases to consider (all other cases are either symmetric, or have a near-identical argument).

$T_{s_{t_1}} \neq \bullet \neq T_{s_{t_2}}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (TRY-OK$_\Rightarrow$).
So $\sigma'_j = \sigma_{s_{t_j}}$.
Thus vii) and viii) follow from reflexivity and transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.
Since $R_j = T_{s_{t_j}} \neq \bullet$, vi) holds vacuously.

$T_{s_{t_1}} = \bullet = T_{s_{t_2}}$**:** Then iv), for both $j = 1$ and $j = 2$, was established through (TRY-E$_\Rightarrow$).
In both runs, $s_c$ is evaluated.
So $\langle \sigma_{s_{t_j}}, s_c \rangle \Rightarrow \langle \sigma'_j, R_j \rangle$.
By (IH)$_s$, $R_j \neq \bullet = R_{\bar{j}} \implies \ell' \not\sqsubseteq \ell$, $\sigma'_1 =_\ell \sigma'_2$ and $\sigma'_1 \sim_\ell^{\Gamma_c} \sigma'_2$, so vi), vii) and viii) hold.

$T_{s_{t_1}} \neq \bullet = T_{s_{t_2}}$**:** Then iv), for $j = 1$, resp. $j = 2$, was established through (TRY-OK$_\Rightarrow$), resp. (TRY-E$_\Rightarrow$).
So, $\sigma'_1 = \sigma_{s_{t_1}}$, $R_1 = T_1 = \texttt{skip}$ and $\langle \sigma_{s_{t_2}}, s_c \rangle \Rightarrow \langle \sigma'_2, R_2 \rangle$.
Since $T_{s_{t_1}} \neq \bullet = T_{s_{t_2}}$, $\ell_t \not\sqsubseteq \ell$.
So by Lemma 7.6, if $R_2 = \bullet$, $\ell' \not\sqsubseteq \ell$. So vi) holds.
As for $j = 1$, since $s_c$ is evaluated under context $pc \sqcup \ell_t$, we get from Lemmas 7.7 and 7.7 that $\sigma_{s_{t_1}} \sim_\ell^{\Gamma_c} \sigma'_1$ and $\sigma_{s_{t_1}} =_\ell \sigma'_1$.
Since $\Gamma_t^{\mathsf{c}} \sqsubseteq \Gamma_c^{\mathsf{c}}$ and $\Gamma_t^{\mathsf{e}} \sqsubseteq \Gamma_c^{\mathsf{e}}$, $\sigma_{s_{t_1}} \sim_\ell^{\Gamma'} \sigma'_1$ holds.
So vii) and viii) both hold by reflexicity and transitivity of $\sim_\ell^{\Gamma'}$ and $=_\ell$.

**(**WHILE$_\vdash$**)** : Then $\hat{s} = \texttt{while}\ e\ \texttt{do}\ s$, for some $e$ and $s$.
**Well-typing of arbitrarily long $es$-sequence:**
By *i)*, $pc_i^e \vdash \hat{\Gamma}_i\,\{e\}\,\hat{\Gamma}'_i : \ell_i^e$ and $pc_i^s \vdash \hat{\Gamma}'_i\,\{s\}\,\hat{\Gamma}_{i+1} : \ell_i^s$,
where $pc_i^e = pc \sqcup \ell'_i$, $pc_i^s = pc \sqcup \ell'_i \sqcup \ell_i^e \sqcup \mathsf{lvl}(e)$,
$\ell'_0 = \bot$, $\ell'_{i+1} = \ell'_i \sqcup \ell_i^e \sqcup \ell_i^s$,
$i = 0..n$, $(\hat{\Gamma}_n, \ell'_n) = (\hat{\Gamma}_{n+1}, \ell'_{n+1})$,
$\ell' = \ell'_n$, $\Gamma = \hat{\Gamma}_0$ and $\Gamma' = \bigodot_{j=0}^{n} \hat{\Gamma}'_j \odot \hat{\Gamma}_{j+1}$.
From this, and since the type system is deterministic,
we have for all $k' > n$ that $(\hat{\Gamma}_{k'}, \ell'_{k'}) = (\hat{\Gamma}_{k'-1}, \ell'_{k'-1})$.
By transitivity, $(\hat{\Gamma}_{k'}, \ell'_{k'}) = (\hat{\Gamma}_n, \ell'_n)$.
So $pc_{k'}^e \vdash \hat{\Gamma}_{k'}\,\{e\}\,\hat{\Gamma}'_{k'} : \ell_{k'}^e$ and $pc_{k'}^s \vdash \hat{\Gamma}'_{k'}\,\{s\}\,\hat{\Gamma}_{k'+1} : \ell_{k'}^s$, for $k' \geq 0$.
By Lemma 7.3, for all $k' \geq 0$, $\vdash_{\mathsf{dep}} \Gamma'_k$ and $\vdash_{\mathsf{dep}} \Gamma_{k+1}$.
**Notation for sequence of $es$ evaluations:**

Let $j = 1..2$ and

$$
\begin{aligned}
t_k &= \begin{cases} e & \text{, if } k \text{ is even} \\ s & \text{, otherwise} \end{cases} \\
\sigma_{j0} &= \sigma_{j0} \\
\langle \sigma_{jk}, t_k \rangle &\Rightarrow \langle \sigma_{jk}', R_{jk} \rangle \\
\sigma_{jk+1} &= \sigma_{jk}' \\
T_{jk} &= \begin{cases} \bullet & \text{, if } R_{jk} = \bullet \\ \texttt{skip} & \text{, if } R_{jk} = 0 \end{cases} \\
pc_k &= \begin{cases} pc^e_{k/2} & \text{, if } k \text{ is even} \\ pc^s_{k-1/2} & \text{, otherwise} \end{cases} \\
\ell_k &= \begin{cases} \ell^e_{k/2} & \text{, if } k \text{ is even} \\ \ell^s_{k-1/2} & \text{, otherwise} \end{cases} \\
\Gamma_k &= \begin{cases} \hat{\Gamma}_{k/2} & \text{, if } k \text{ is even} \\ \hat{\Gamma}'_{k-1/2} & \text{, otherwise} \end{cases} \\
\Gamma'_k &= \Gamma_{k+1}
\end{aligned}
$$

Then $pc_k \vdash \Gamma_k \{t_k\} \Gamma'_k : \ell_k$.
Also, $\ell_k \sqsubseteq pc_{\hat{k}}$, for all $\hat{k} > k$.
Furthermore, $\ell_k \sqsubseteq \ell'$, for all $k$.
**Pairwise memory equivalence:**
Like in the proof of Lemma 7.7 for $s$, we have that, for both $j$,
then $\sigma'_j = \sigma_{jk}'$ for the least $k$ for which $T_{jk}$ is defined.
Note that $T_{jk}$ is defined for at least one value of $k$, by *v)*.
To establish $\sigma'_1 =_\ell \sigma'_2$ and $\sigma'_1 \sim_\ell \sigma'_2$, we first prove the following, for all $k$, assuming $\sigma_{1k} =_\ell \sigma_{2k}$ and $\sigma_{1k} \sim_\ell \sigma_{2k}$.

$$\sigma_{1k+1} =_\ell \sigma_{2k+1} \wedge \sigma_{1k+1} \sim_\ell \sigma_{2k+1} \tag{3}$$

To establish this, we will need to use

$$pc_k \vdash \Gamma_k \{t_k\} \Gamma'_k : \ell_k \quad \vdash_{\mathsf{dep}} \Gamma_k \quad \vdash_{\mathsf{dep}} \sigma_{jk} \quad \Gamma_k \models_{\mathsf{dep}} \sigma_{jk} \quad \vdash_{\mathsf{err}} \sigma_{jk} \tag{1}$$

We have already established $pc_k \vdash \Gamma_k \{t_k\} \Gamma'_k : \ell_k$ and $\vdash_{\mathsf{dep}} \Gamma_k$ for all $k$.
We now establish $\vdash_{\mathsf{dep}} \sigma_{jk}$, $\Gamma_k \models_{\mathsf{dep}} \sigma_{jk}$ and $\vdash_{\mathsf{err}} \sigma_{jk}$ for all $k$.
So, with $k$ arbitrary, assuming (1), we must show

$$\vdash_{\mathsf{dep}} \sigma_{jk+1} \quad \Gamma_{k+1} \models_{\mathsf{dep}} \sigma_{jk+1} \quad \vdash_{\mathsf{err}} \sigma_{jk+1} \tag{2}$$

(2) follows directly from Lemmas 7.4 and 7.5.
We now prove (3), for all $k$.
With $k$ arbitrary, assume $\sigma_{1k} =_\ell \sigma_{2k}$, $\sigma_{1k} \sim_\ell \sigma_{2k}$ and (1).
By Lemma 7.8 for $a$ (in case $t_k = e$), or (IH) (in case $t_k = s$),
$\sigma_{1k+1} =_\ell \sigma_{2k+1}$ and $\sigma_{1k+1} \sim_\ell \sigma_{2k+1}$.
So (3) holds.
**Component-wise memory equivalence under $\not\sqsubseteq \ell$ contexts:**
Another result that we will need is the following.

$$pc_k \not\sqsubseteq \ell \implies \sigma_{jk} =_\ell \sigma_{jk+1} \wedge \sigma_{jk} \sim_\ell \sigma_{jk+1} \tag{4}$$

Assume (1).
Assume $pc_k \not\sqsubseteq \ell$.
By Lemma 7.7,
$\sigma_{jk} =_\ell \sigma_{jk+1}$ and $\sigma_{jk} \sim_\ell \sigma_{jk+1}$.
So (4) holds.
**Towards *vi)*, *vii)* and *viii)*:**
By Lemma 7.8 for $a$,
$\sigma_{11} =_\ell \sigma_{21}$ and $\sigma_{11} \sim_\ell \sigma_{21}$.
By Lemma A.8, $\sigma_{jk} = \sigma_{jk}' = \sigma_{jk+1}$ for even $k$.

Let $k_j$ be least such that $T_{j\,k_j}$ is defined.
Assume wlg. that $k_1 \leq k_2$.
Two cases to consider:

$t_{k_1} = s$**:** Then $R_{1k_1} = \bullet$; else contradicting the definition of $T_{1k_1}$.
Case on $R_{2k_1}$

$R_{2k_1} = \bullet$**:** Then $k_1 = k_2 =: k$ and $\sigma'_j = \sigma_{j\,k}{}'$.
So *vii)* and *viii)* hold, and *vi)* holds vacuously.

$R_{2k_1} = \texttt{skip}$**:** Then by (IH), $\ell_{k_1} \not\sqsubseteq \ell$.
Since $\ell_{k_1} \sqsubseteq \ell'$, $\ell' \not\sqsubseteq \ell$.
So *vi)* holds (regardless of whether $R_1$ and $R_2$ differ or not).
For all $\hat{k} > k_1$, since $\ell_{k_1} \sqsubseteq pc_{\hat{k}}$, $pc_{\hat{k}} \not\sqsubseteq \ell$.
By (4), $\sigma_{2k_1} =_\ell \sigma_{2k_2}$ and $\sigma_{2k_1} \sim_\ell \sigma_{2k_2}$.
Thus $\sigma_{1k_1} =_\ell \sigma_{2k_2}$ and $\sigma_{1k_1} \sim_\ell \sigma_{2k_2}$.
So *vii)* and *viii)* hold.

$t_{k_1} = e$**:** Then $R_{1k_1} \in \{0, \bullet\}$.
We prove the $R_{1k_1} = 0$ case, since the proof of the $R_{1k_1} = \bullet$ case is obtained by swapping the proofs of the first two cases in the following case distinctions on $R_{2k_1}$ and $R_{2k_2}$.
Case on $R_{2k_1}$.

$R_{2k_1} = 0$**:** Then $k_1 = k_2 =: k$ and $\sigma'_j = \sigma_{j\,k}{}'$.
So *vii)* and *viii)* hold, and *vi)* holds vacuously.

$R_{2k_1} = \bullet$ Then $k_1 = k_2 =: k$ and $\sigma'_j = \sigma_{j\,k}{}'$.
So *vii)* and *viii)* hold.
We also have $\ell_k \not\sqsubseteq \ell$.
Since $\ell_k \sqsubseteq \ell'$, we get $\ell' \not\sqsubseteq \ell$.
So *vi)* holds.

$R_{2k_1} \notin \{0, \bullet\}$**:** Then $\mathsf{lvl}(e) \not\sqsubseteq \ell$; else contradicting $\sigma_{1k_1} =_\ell \sigma_{2k_1}$.
Since $R_{1k_1} = 0$, $t_{k_1} = e$, so $k_1$ is even.
Let $\hat{k}$ range over $k_1 + 1 + 2m$, for nonnegative integer $m$.
(So $\hat{k}$ ranges over all odd integers $\geq k_1$).
Then $t_{\hat{k}} = s$, for all $\hat{k}$.
Since $\mathsf{lvl}(e) \sqsubseteq pc_{\hat{k}}$, we get by (4) that $\sigma_{2\hat{k}} =_\ell \sigma_{2\hat{k}+1}$ and $\sigma_{2\hat{k}} \sim_\ell \sigma_{2\hat{k}+1}$.
Since we already have by Lemma A.8 that $\sigma_{j\,k} = \sigma_{j\,k}{}' = \sigma_{j\,k+1}$ for even $k$,
we get $\sigma_{2k_1} =_\ell \sigma_{2k_2}$ and $\sigma_{2k_1} \sim_\ell \sigma_{2k_2}$.
By transitivity, $\sigma_{1k_1} =_\ell \sigma_{2k_2}$ and $\sigma_{1k_1} \sim_\ell \sigma_{2k_2}$.
So *vii)* and *viii)* hold.
Case on $t_{k_2}$.

$t_{k_2} = e$**:** Case on $R_{2k_2}$.

$R_{2k_2} = 0$**:** Then *vi)* holds vacuously.

$R_{2k_2} = \bullet$**:** Then, since $\sigma_{2k_1} =_\ell \sigma_{2k_2}$ and $\sigma_{2k_1} \sim_\ell \sigma_{2k_2}$,
since $\sigma_{2k_1} \sqsubseteq \sigma_{2k_2}$,
and thus since $\Gamma_{k_1} \models_{\mathsf{dep}} \sigma_{2k_2}$,
we get by Lemma 7.8 for $a$ that $\ell_{k_1} \not\sqsubseteq \ell$.
Since $\ell_{k_1} \sqsubseteq \ell'$, $\ell' \not\sqsubseteq \ell$.
So *vi)* holds.

$t_{k_2} = s$**:** Then $R_{2k_2} = \bullet$; else contradicting that $T_{2k_2}$ is defined.
Since $\mathsf{lvl}(e) \sqsubseteq pc_{k_2}$ and $\mathsf{lvl}(e) \not\sqsubseteq \ell$,
we get by Lemma 7.7 that $\ell_{k_2} \not\sqsubseteq \ell$.
Since $\ell_{k_2} \sqsubseteq \ell'$, $\ell' \not\sqsubseteq \ell$.
So *vi)* holds.

This concludes the proof.

□